On Some Theories On Quantum Space-Time And Matter And Their Plausible Implications

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Dedicated to my parents and sisters.....

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List of publications

1. *"Connecting dissipation and noncommutativity: A Bateman system case study"*

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4. *"Particle dynamics and Lie-algebraic type of non-commutativity of space–time"*

   Partha Nandi, **Sayan Kumar Pal**, Ravikant Verma

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5. *"Revisiting quantum mechanics on non-commutative space-time"*

   Partha Nandi, **Sayan Kumar Pal**, Aritra N. Bose, Biswajit Chakraborty

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6. *"A note on broken dilatation symmetry in planar noncommutative theory"*

   Partha Nandi, Sankarshan Sahu, **Sayan Kumar Pal**

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7. *"Fuzzy Classical Dynamics as a Paradigm for Emerging Lorentz Geometries"*

   F.G. Scholtz, Partha Nandi, **Sayan Kumar Pal**, Biswajit Chakraborty

   e-print: arXiv:1905.05018 [hep-th].

*This thesis is based only on the papers 1,2,3,4.

Contents

Chapter 1

Introduction

A complete theory of quantum gravity remains quite elusive even to this day, despite numerous attempts made by a large number of physicists across the globe. And among all of the approaches towards this goal, which the physicists have undertaken so far, the approach through String theory remains most popular, in the sense that a huge number of physicists are working along this line in comparison to other approaches like Loop quantum gravity, Noncommutative geometry, etc. But despite all the efforts that have been invested in String theory, the practitioners of this theory have failed to come up with a convincing prediction which can perhaps be confronted with experiment. In this sense, it is not even falsifiable! In contrast, in a completely bottom-up approach, Alain Connes and his other collaborators have developed [1–3] a completely new paradigm for the Standard model of particle physics, where they showed how a completely novel framework for a unified theory of Einstein gravity and the gauge theory of the Standard model can be formulated by making use of the mathematical framework of Noncommutative geometry, developed earlier by Connes himself [4]. In this framework, even the Higgs field gets unified with other gauge fields, in the sense that the Higgs field can themselves be viewed as sort of connections in the discrete directions, described by certain matrix algebra. And this perspective, in turn, made it possible for Connes et. al. to compute the correct Higgs mass eventually [2]. But again, despite many other such successes, a main shortcoming of this framework is that the above mentioned unification holds at the classical level only and the energy scale associated with this is much below the GUT scale, which in turn, is couple of orders below the Planck scale. A direct extrapolation of this to a scale, which is in the vicinity of Planck scale will not be a sensible thing to do, as it is quite likely that the nature of space-time itself changes dramatically at that scale and is no-longer given by a pseudo-Riemannian differentiable manifold. In fact, a simple extrapolation of general relativity and quantum theory to this scale indicates that a localization of an event down to this scale is virtually impossible; the very process will help form a black hole with an attendant event horizon, thereby the trapping all the information contained therein.

In fact, this possibility was pointed out by Bronstein way back in 1930's or so. And more recently a plausible scenario was proposed, in a fairly rigorous approach, by Doplicher et.al. [5, 6], where the space-time coordinates themselves are promoted to the level of operators, satisfying suitable non-vanishing coordinate algebra, so that such a gravitational collapse can perhaps be avoided, as the coordinates are now subject to uncertainty relations a la Heisenberg. Clearly, in this scenario the space-time is now devoid of any point-like structures and becomes fuzzy-so to speak. This was also corroborated later through studies in certain low energy effects of string theory by Seiberg and Witten [7]. These developments triggered an upsurge of interest, where various authors tried to formulate Quantum mechanics (QM) and Quantum Field Theory (QFT) on the background of such fuzzy spaces and tried to look for any footprints of noncommutativity, which in turn can be regarded as a sort of quantum gravity effect. In all these developments, the basic idea has been to capture the quantum effects of space-time (quantum space-time) by postulating a noncommutative algebra between space-time which are being promoted to quantum operators now. Through this, one is able to incorporate a fundamental minimal length scale in these theories and is therefore intrinsically consistent for capturing minimal length scale physics. In this context, it will be worthwhile to mention that the spread of the wave function, or the probability density for that matter, has been shown to be greater than the natural length scale $\sim \sqrt{\theta}$ and cannot be squeezed below this scale even in the limit of infinite confining potential [8], implying that noncommutativity may play a role in preventing the formation of a black hole. Noncommutativity can also be considered as a deformation of the standard Heisenberg-Weyl algebra using the notion of quantum groups [9].

It should, however, be noted that the conjectured noncommutative parameters are expected to be extremely tiny i.e. of the order associated to the Planck length scale ($l_p \approx 10^{-33} cm$). Consequently, any possible observation of such effects through terrestrial experiments remain highly unlikely. The best one can do, at this stage, is to try develop theoretical models to see whether some of the expected features of quantum gravity can be accommodated there naturally or not. And it is in this respect that one is reminded of the well known 't Hooft conjecture [10, 11], where he argued about dissipative effects and the resulting information loss i.e. entropy production at the quantum gravity scale. To solve the black hole information paradox, he proposed a new scheme for quantizing Hamiltonians having the form, $H = \sum_i p_i f_i(\vec{q})$ which are not bounded from below, resulting in no straightforward canonical quantization. This investigation led to the hypothesis that the dynamics of quantum fields near Planck scales may be governed by viscosity itself. Dissipation seems to be inevitable and all-important at very very high energy scales. Very recently, this scheme has been adopted for quantizing Bateman's doubled system used by us for describing the damped harmonic oscillator (DHO) and dissipative dynamics [12]. It is interesting to mention, in this context, that noncommutative or deformed systems have been shown to violate time-reversal symmetry [13], which is also the case with dissipative dynamics.

Furthermore, apart from the above mentioned noncommutativity among the spatio-temporal coordinates, one can also expect momentum space noncommutativity as well, as follows from the celebrated "reciprocity theorem" due to Max Born [14]. Momentum space noncommutativity is relatively a more familiar set of noncommutative structure as compared with position space noncommutativity in the sense that it is well-known that the translation generators or the momenta do not commute in the case of a curved space. Conversely, from a similar point of view, curved momentum spaces give rise to noncommutativity in the position coordinates. Thus, Born suggested that in order to combine general relativity with quantum mechanics, one needs also to consider curved momentum spaces in an analogous manner as follows from the principles of quantum mechanics (principle of reciprocity) i.e. as a Riemannian structure, both the coordinate space and the momentum space should have similar geometrical structures and status. Indeed, this principle has been effectively used to study Planck scale physics using Hopf algebras in [9, 15], where it has been inferred that quantum phase-space basically incorporates quantum spacetime and that noncommutativity should exist in both position and momentum variables in order to represent a quantum gravity theory. Furthermore, the co-existence of position and momentum space noncommutativity has been demonstrated in condensed matter physics for some planar systems, albeit with much larger scales of the corresponding NC parameters. Indeed it has been found that certain crystals exhibiting Berry curvature in momentum space in presence of an applied magnetic field do manifest phase space noncommutativity [16]. In current theory of magnetization, the accompanying transformation of the Berry phase formula in terms of Wannier functions provides an alternate and intuitive viewpoint of electric polarisation, and helps explain the anomalous Hall effect [17, 18]. This line of study has been supported by many experimental observations [19]. Therefore, Berry phase dynamics leads to a non-standard (deformed) symplectic form in an effective setting. Also, the spatial noncommutativity parameter (θ), for the NC Moyal plane: $[\hat{x}_i, \hat{x}_j] = i\theta\epsilon_{ij}$, has been shown [20, 21] to be related to the anyonic spin i.e. fractional (arbitrary) spin of anyons, which can only appear as excitations in two spatial dimensions [22, 23] and have been found to be extremely useful in the description of fractional quantum Hall effect (FQHE). This stems from a peculiar nature of the Galilean group in 2+1 dimensions [24], in particular we have, $[K_i, P_j] = i\hbar m\delta_{ij}$ and $[K_i, K_j] = i\hbar s\epsilon_{ij}$, where K_i and P_i are boost and translation generators respectively, m is the mass and s is the nonrelativistic arbitrary spin parameter obtained from a suitable Inonu-Wigner group contraction of the relativistic Poincare group in 2+1 dimensions [25]. Thus it admits a two-fold central extension - one is the mass and the second extension corresponds to the above mentioned anyonic spin s, which is an arbitrary real number. This latter central extension is the exotic one and due to this peculiarity of the planar Galilei group, the corresponding Lagrangian one-form needs to be augmented with an extra term leading to a deformed symplectic structure [20, 21, 25]. Therefore in 2+1 dimensions, which will be the setting for Chapters 2 and 3 of this thesis, deformation or spatial noncommutativity can also be attributed with the second-central extension of the planar Galilei group or the fractional spin of particles (the anyonic spin). With the recent experimental detection of anyons [26] in a particular heterogeneous semiconductor, a new wave of interest has been aroused among theoreticians to wander more on these exotic particles. On the other hand, it is well-known now that the widely used guiding-center coordinates in the lowest Landau level also exhibit noncommutative algebra, as well as upon projection of the system to lowest Landau level or a few number of Landau levels, the coordinates also satisfy NC algebra [27]. Therefore, it is not surprising that Noncommutative geometry, being developed over the years, has found much relevance in the study of the integer and the fractional quantum Hall effect (FQHE) [28–30]. Moreover in the physics of cold Rydberg atoms, it has been suggested that spatial NC emerges quite naturally [31, 32] and NC physics also appears in the description of the chiral magnetic effect observed recently in Fermi liquids [33] and, in skew-scattering in ferromagnetic metals [34] owing to the presence of Berry curvature in momentum space of the Bloch wavefunctions mentioned above. All these developments thus signify that, apart from Planck scale physics, noncommutativity (deformation in symplectic structure) has emerged as a key aspect in modern condensed matter theories also.

On the other hand, t' Hooft's scheme of quantization [10] also gives rise to a deformation in the symplectic structure between the coordinates, thus a noncommutative geometry is obtained [35] in this scheme as well. However here, the quantization is only achieved as a consequence of information loss in the system. The final system turns out to have lower bound and is integrable. For a quantum system, the von-Neumann entropy provides a measure of information loss associated with the system. In present day physics, the concept of entropy has turned out to play a key role to make our understanding of the universe more clear. Computation of the entropy function and analysing its properties have revealed intriguing physics aspects in different systems such as Unruh problem, dissipative systems, etc. We plan to investigate entropic aspects of deformed theories in the thesis.

There have been many approaches to study quantum systems in the framework of noncommutative geometry. The most traditional one is the Weyl-Wigner correspondence involving Moyal star products [36, 37]. Here one works with commutative functions instead of noncommutative operators and the noncommutative effects are captured by star products among the functions [38]. Second method is the Hilbert-Schmidt operator formulation working with noncommutative operators ab initio and this involves working with maximally localized position states (coherent states) and the Voros star product [13,39,40]. The other approach is to convert the original noncommutative theory into an effective commutative theory at the level of phase-space variables (operators) by making use of suitable transformations, Bopp's shift which generally leads to deformation of the system Hamiltonian and the symplectic structure reduces to the standard one [41–43].

Part of this present thesis primarily aims to study these two aspects, i.e. dissipation and entropy production, as mentioned above in the planar noncommutative setting i.e. the Moyal plane. Our study will bring out interesting aspects of deformed theories as we shall see in the course of this thesis. We shall be mainly focusing our attention to some simple quantum mechanical models having spatial and/or momentum space noncommutativity and analyse these different models by comparing with their commutative counterparts. We will also try to model the emerging characteristics/phenomena encountered in our studies in known common systems of interest. Also we discuss effects of deformation on macroscopic quantities such as pressure, mass of a macroscopic object, etc., which will help to serve as a viable model to explain the enhanced Chandrasekhar mass limit of certain over-luminous supernovae observed recently [44, 45]. Lastly, we will consider deformations between space and time, meaning that there would be a noncommutative algebra only between space and time coordinates. In contrast to the Moyal-like noncommutativity which involves a constant noncommutative parameter, an example of a space-time noncommutative structure that has attracted much attention is the Kappa-Minkowski space-time [46] and this space-time will be the subject of our study as an interesting case of space-time noncommutativity. It involves a Lie-algebraic kind of deformation of ordinary commutative space-time algebra.

1.0.1 Organization of the thesis:

This thesis is based mainly on the following works [47–50] and it has been organized into six chapters, where chapter 1 consists of the present "Introduction" to the thesis. The outlines of the subsequent chapters are as follows:
In chapter 2, we discuss dissipative dynamics in two-dimensional noncommutative Moyal plane by using Bateman oscillators [51]. The system Hamiltonian does not have a lower bound and we overcome this issue by incorporating extra interactions in the Bateman system. We carry out path-integral quantization of this augmented system in Moyal plane using coherent states in the framework of Hilbert-Schmidt operators. Subsequently, we analyse the Bateman system and observe that noncommutativity can act as a source of dissipation.

In chapter 3, we analyse entropic aspects of deformed systems such as the harmonic oscillators in Moyal plane as well as in phase-space noncommutative plane. We find that a non-zero entanglement entropy proportional to the noncommutative parameter/s appears in such systems. By establishing a correspondence of noncommutative (exotic) oscillators with the Landau problem in presence of harmonic potential, we describe a Unruh-like effect. Also, we provide a discussion of squeezing phenomenon in non-linear crystals which has a surprising mathematical resemblance with the present problem.

In chapter 4, we take up momentum noncommutativity and discuss its effect on the spectrum of a three-dimensional free electron gas. We then study the quantum mechanical statistical properties of the gas in our case and we find that there is an increase in the degeneracy pressure. An implication is then readily sought for in the case of white dwarf stars. We find that the critical mass-limit of these stars is higher in such a scenario.

In chapter 5, we consider the case of space-time noncommutativity, in particular Kappa-Minkowski space-time. We will derive the Kappa-Minkowski commutation relation in mechanical systems by converting the action into a time reparametrization-invariant form. Subsequently, the noncommutativity will emerge in the form of Dirac brackets on carrying out the constraint analysis. Our method provides a dynamical realization of Kappa-Minkowski space-time algebra in both relativistic and nonrelativistic contexts.

The thesis ends in chapter 6 with certain concluding remarks and future plans.

Chapter 2

Dissipation in noncommutative spaces

The damped harmonic oscillator (DHO) problem has been quite ubiquitous in physics, appearing in a large variety of physical situations. This problem is also characterised by the breaking of time-reversal symmetry. On the other hand, noncommutative dynamics has also been known to violate time-reversal symmetry, see for example [13]. Furthermore, dissipation is believed to play a dominant role in the dynamics of quantum fields in the vicinity of Planck regime [10, 52–54]. In fact, it is exactly in these scales that one expects a complete breakdown of known laws of physics and the concept of space-time, modelled as pseudo-Riemannian differentiable manifold. Indeed, as per some strong plausibility arguments by Doplicher et. al., the differentiable manifold model of spacetime should be replaced by the regime of quantum space-time theories which, in turn, may be modelled by some noncommutative algebra, where one postulates non-vanishing commutators among the coordinates, which are now promoted to the level of operators. In this chapter, we venture out to see the nature of dissipative theories in noncommutative setting and find out any interplay between these two previously unrelated aspects. Indeed there have been some works in the literature previously which sowed the seeds of connection between these aspects [10]. Having said this, we describe the background of the problem in the sequel to follow.

The DHO problem does not lend itself to a direct Lagrangian formulation, preventing a subsequent standard canonical analysis because it leads to explicitly time dependent Lagrangians [55]. However, by augmenting the DHO with its time reversed image, H. Bateman [51] developed a time-independent Lagrangian for the damped harmonic oscillator (DHO) problem in the 1930s. In this formulation, one works with an effective doubled system. The latter system corresponds to the so-called "anti-damped" oscillator. Although this allows one to write a time-independent Lagrangian (Eq. (2.5)) describing the system, the quantization of such a system has been a non-trivial task. This is because of the fact that the Legendre-transformed Hamiltonian of the system turns out to have no lower bound. We provide here a novel approach to the problem of quantization of the damped harmonic oscillator from the ones existing in the literature and we take the ambient space to be noncommutative providing a generalization to a deformed scenario as well. However, such systems can be quantized, in some cases like this, by adopting 't Hooft's scheme [10], as shown in [12] by working on a suitable physical subspace where the Hamiltonian is bounded from below. This scheme of quantization [10] was provided for Hamiltonians of the form $H = \sum_i p_i \, f_i(\boldsymbol{q})$, where $f_i(\mathbf{q})$ are non–singular functions of the canonical coordinates q_i. Quantization is only achieved in this technique as a result of dissipation of information. This group of Hamiltonians isn't usually bounded from below. This could be fixed by seeking for a time independent positive function $\rho(\mathbf{q})$ that satisfies $\{\rho, H\} = 0$. In that case, we essentially have $H|\psi_{ph} >= \rho(\boldsymbol{q})|\psi_{ph} >$. Furthermore, this scheme of quantization leads to a deformation in the symplectic structure as can be understood from the Hamilton's equation of motion

$$\dot{q}_i = f_i(\mathbf{q}) = \{q_i, \rho(\mathbf{q})\}$$

Clearly this indicates that the Poisson brackets (or the corresponding commutator at the operator level) among the q_i's must be non-zero thus leading to noncommutative structures in order to ensure non-vanishing velocities ($\dot{\mathbf{q}}$) in the general case. This situation is somewhat similar to the well-known Landau problem of a charged particle moving in a plane and subjected to a constant magnetic field normal to the plane, in the sense that the original commuting coordinate operators in the Landau problem become noncommutative when projected to the subspace associated with lowest Landau level [27]. Here in this thesis, we see how a consistent quantization can be formulated in noncommutative Moyal plane by considering additional interactions.

This chapter has been arranged as follows: We describe the problem in detail and will reformulate it by including extra interactions to be discussed next. Then we move to section 2.1 which contains a brief discussion of all the machinery needed to carry out the analysis in noncommutative scenario. In section 2.2, main investigation of dissipative dynamics in Moyal plane is carried out using coherent state path-integral, followed by canonical analysis in section 2.3. In section 2.4, the reduction to Bateman system is demonstrated and lastly in section 2.5 concluding remarks has been made.

We begin with a discussion of the problem of classical damped harmonic oscillator (DHO). The DHO is represented by the following equation of motion -

$$\ddot{x} + \gamma \dot{x} + \omega^2 x = 0 \tag{2.1}$$

Following Bateman's doubling method [51], this system is to be augmented by its time-reversed equation of motion -

$$\ddot{y} - \gamma \dot{y} + \omega^2 y = 0 \tag{2.2}$$

Now, to obtain the action which will yield the above pair of equations under extremization, one has to solve the inverse variational problem. This can be achieved by seeking for an action such that its infinitesimal variation takes the following form:

$$\delta S = -\int_{t_1}^{t_2} dt \left[\left(\ddot{x} + \gamma \dot{x} + \omega^2 x \right) \delta y + \left(\ddot{y} - \gamma \dot{y} + \omega^2 y \right) \delta x \right] \tag{2.3}$$

where it is evident that variation w.r.t. x and y automatically yields (Eq. (2.2)) and (Eq. (2.1)) respectively with the usual boundary conditions where δx and δy vanishes at t_1 and t_2. Now notice that it is feasible to transform it into an integrable form by employing integration by parts and using the standard boundary conditions as,

$$\delta S = \delta \int_{t_1}^{t_2} dt \left[\dot{x}\dot{y} + \frac{\gamma}{2}\left(x\dot{y} - \dot{x}y\right) - \omega^2 xy\right] \tag{2.4}$$

Reading off easily the action from the above equation, we have therefore

$$S = \int dtL \quad ; \quad L = \dot{x}\dot{y} + \frac{\gamma}{2}\left(x\dot{y} - \dot{x}y\right) - \omega^2 xy \tag{2.5}$$

Here the parameters $\gamma > 0$ and ω are independent of time representing the damping parameter and the frequency respectively. Here, the mass has been taken as unity. This Lagrangian has been referred as an "indirect Lagrangian" in the literature [56, 57].

Now considering DHO equation (2.1), we can set $x(t) = e^{i\lambda t}$, as an ansatz, to find the characteristic frequency λ as,

$$\lambda_{\pm} = \pm i\frac{\gamma}{2} + \frac{\gamma}{2}\sqrt{R^2 - 1} \tag{2.6}$$

where the critical ratio $R = \frac{2\omega}{\gamma}$ determines the dynamics of the system. If $R > 1$, the system given by (Eq. (2.1)) has an oscillatory motion with exponentially decaying amplitude. The motion is otherwise non-oscillatory and overdamped.

Now, we introduce the so-called light-cone coordinates x_1 and x_2 by using the transformation T_1 given as,

$$T_1 : \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.7}$$

then the Lagrangian (Eq. (2.5)) can be re-written in a standard quadratic form as,

$$L = \frac{1}{2} g_{ij}\dot{x}_i\dot{x}_j - \frac{\gamma}{2}\epsilon_{ij}x_i\dot{x}_j - \frac{\omega^2}{2} g_{ij}x_i x_j \tag{2.8}$$

[1] where we take note of a strange appearance of a pseudo-Euclidean (Lorentzian) spatial metric g_{ij}; g_{11} = -g_{22} = 1 and $g_{12} = 0$.

The Hamiltonian corresponding to (Eq. (2.8)) is given by-

$$H = H_1 - H_2 \tag{2.9}$$

where

$$H_1 = \frac{1}{2}(p_1 - \frac{\gamma x_2}{2})^2 + \frac{1}{2}\omega^2 x_1^2 \ , \ H_2 = \frac{1}{2}(p_2 + \frac{\gamma x_1}{2})^2 + \frac{1}{2}\omega^2 x_2^2 \tag{2.10}$$

Therefore the Hamiltonian is a difference of two positive definite Hamiltonians such that it is not bounded from below. To address this issue, we consider additional interactions to (Eq. (2.1)) and (Eq. (2.2)) in order to make the Hamiltonian positive, which will be accomplished in part by introducing linear and second-order derivative couplings between 'x' and 'y' oscillators with strength parameters of ϵ and η, respectively. Thus instead of (Eq. (2.1)) and (Eq. (2.2)) we introduce the following pair of "master equations" -

$$\ddot{x} + \gamma\dot{x} + \omega^2 x = -\epsilon y - \eta\ddot{y} \tag{2.11}$$

$$\ddot{y} - \gamma\dot{y} + \omega^2 y = -\epsilon x - \eta\ddot{x} \tag{2.12}$$

This pair of equations now represent an enlarged system of a damped Bateman oscillator pair with an additional 2D harmonic oscillator of mass η and spring constant ϵ . These equations of motion follow from the Lagrangian -

$$L = \dot{x}\dot{y} + \frac{\gamma}{2}\,(x\dot{y} - \dot{x}y) - \omega^2 xy + \frac{\eta}{2}(\dot{x^2} + \dot{y^2}) - \frac{\epsilon}{2}(x^2 + y^2)$$

[1]Except for its Lorentzian signature, the composite Lagrangian (Eq. (2.8)) is structurally similar to the generic two-dimensional Euclidean planar oscillator in a magnetic field, where $g_{ij} = \delta_{ij}$ and γ is the magnetic field.

which when expressed in terms of light cone coordinates Eq. (2.7) takes the following form-

$$L = \frac{(\eta+1)}{2}\dot{x}_1^2 + \frac{(\eta-1)}{2}\dot{x}_2^2 - \frac{\gamma}{2}\left(x_1\dot{x}_2 - x_2\dot{x}_1\right) - \frac{(\epsilon+\omega^2)}{2}x_1^2 - \frac{(\epsilon-\omega^2)}{2}x_2^2 \tag{2.13}$$

The linearly interacting case of the Bateman modes has been considered very recently in [58] to explain a system of two coupled whispering-gallery-mode optical resonators [59] whereas the introduction of second-order derivative couplings along with a linear coupling is completely new. It is worthwhile to mention here that both of these couplings will be critical for a consistent quantum mechanical formulation. The corresponding Hamiltonian in the light-cone coordinates is given by:

$$H = \frac{p_1^2}{2(\eta+1)} + \frac{p_2^2}{2(\eta-1)} + \frac{\gamma}{2}\left(\frac{x_1p_2}{\eta-1} - \frac{x_2p_1}{\eta+1}\right) + \left(\frac{\gamma^2}{8(\eta-1)} + \frac{(\epsilon+\omega^2)}{2}\right)x_1^2 + \left(\frac{\gamma^2}{8(\eta+1)} + \frac{(\epsilon-\omega^2)}{2}\right)x_2^2 \tag{2.14}$$

As one can easily notice from the structure of this equation that the system is no longer isotropic and in this context, we would like to mention that the anisotropic oscillator model, by itself, has many current applications. We can now guarantee the Hamiltonian's positive definiteness by requiring $\eta > 1$ and $\epsilon > \omega^2$. Because the two modes have different effective masses, this form of the Hamiltonian is still not well suited for carrying on the path-integral analysis of the problem. To make it more usable, we'll use the second canonical transformation T_2, which is given by:

$$(x_1, x_2, p_1, p_2)^T \longrightarrow (x_1', x_2', p_1', p_2')^T = T_2(x_1, x_2, p_1, p_2)^T \tag{2.15}$$

$$where, \qquad T_2 = diag\left((\tfrac{\eta+1}{\eta-1})^{\frac{1}{4}},\ (\tfrac{\eta-1}{\eta+1})^{\frac{1}{4}},\ (\tfrac{\eta-1}{\eta+1})^{\frac{1}{4}},\ (\tfrac{\eta+1}{\eta-1})^{\frac{1}{4}}\right). \tag{2.16}$$

so that in terms of the transformed primed variables, the Hamiltonian can be rewritten as,

$$H = \frac{p_1'^2}{2\mu} + \frac{p_2'^2}{2\mu} + \frac{\gamma}{2\mu}(x_1'p_2' - x_2'p_1') + \frac{1}{2}\mu\omega_1^2x_1'^2 + \frac{1}{2}\mu\omega_2^2x_2'^2 \tag{2.17}$$

where the frequencies are given by

$$\omega_1^2 = \frac{\gamma^2}{4(\eta^2-1)} + \frac{\epsilon+\omega^2}{\eta+1} \ \ , \ \omega_2^2 = \frac{\gamma^2}{4(\eta^2-1)} + \frac{\epsilon-\omega^2}{\eta-1} \ \ . \tag{2.18}$$

and the new mass, $\mu = \sqrt{(\eta+1)(\eta-1)}$. For notational brevity, we will drop the primes from the variables while stating the Hamiltonian henceforth. Essentially, this is the final form of Hamiltonian we shall be working with, when we place the system in ambient Moyal plane, where $\hat{x}_1'$ and $\hat{x}_2'$ now satisfy the following noncommutative algebra:

$$[\hat{x}_1', \hat{x}_2'] = [\hat{x}_1, \hat{x}_2] = [\hat{y}, \hat{x}] = i\theta \tag{2.19}$$

Note that in our notation, it is $[\hat{y}, \hat{x}] = i\theta$, rather than $[\hat{x}, \hat{y}] = i\theta$.

2.1 A brief review on HS operator formulation of NCQM

Formulation of quantum mechanics on noncommutative Moyal plane can be achieved through several ways as has been discussed in the literature previously, but the one we are going to adopt in this chapter is based on Hilbert-Schmidt (HS) operator formalism, developed in [13, 39, 60]. Here, we present a brief review of this formalism of noncommutative quantum mechanics. Primarily, this is viewed as a quantum system being represented by an element in the space of Hilbert-Schmidt operators, which act on noncommutative configuration space (Moyal plane) identified as the Hilbert space furnishing a representation of just only the coordinate-coordinate algebra of (Eq. (2.19)).

Formally, the Moyal plane is defined as a 2-D plane, where the pair of coordinate operators satisfy $[\hat{x}_1, \hat{x}_2] = i\theta$ and if a particle moves in the plane then it will be associated with a four-dimensional phase space where the associated operators follow the following noncommutative (deformed) Heisenberg algebra:

$$[\hat{x}_i, \hat{x}_j] = i\theta\epsilon_{ij} \ ; \quad [\hat{p}_i, \hat{p}_j] = 0 \ ; \quad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}. \tag{2.20}$$

Here θ signifies the constant spatial noncommutative parameter that is assumed to be positive ($\theta > 0$) without loss of generality.

The Hilbert space corresponding to noncommutative configuration space is constructed by first observing that this space is isomorphic to the bosonic Fock space of 1D harmonic oscillator (H.O.):

$$\mathcal{H}_c = \text{span}\{|n\rangle = \frac{1}{\sqrt{n!}}(\hat{b}^\dagger)^n|0\rangle\}_{n=0}^{n=\infty} \tag{2.21}$$

where we have now,

$$\hat{b} = \frac{1}{\sqrt{2\theta}}(\hat{x}_1 + i\hat{x}_2); \quad \hat{b}^\dagger = \frac{1}{\sqrt{2\theta}}(\hat{x}_1 - i\hat{x}_2) \tag{2.22}$$

satisfying the Fock algebra $[\hat{b}, \hat{b}^\dagger] = 1$. Note that this $\mathcal{H}_c$ furnishes the representation of just the coordinate algebra (Eq. (2.19)). However, to do quantum mechanics on noncommutative Moyal plane we need a Hilbert space, which furnishes a representation of the entire NC Heisenberg algebra (Eq. (2.20)) which contains (Eq. (2.19)) as a sub-algebra. This is nothing but corresponds to the space $\mathcal{H}_q$ of Hilbert-Schmidt (HS) operators i.e. the space of composite operators $\psi(\hat{x}_1, \hat{x}_2)$, built out of $\hat{x}_1$ and $\hat{x}_2$ having a finite HS norm

$$\mathcal{H}_q = \left\{\psi(\hat{x}_1, \hat{x}_2) : \psi(\hat{x}_1, \hat{x}_2) \in \mathcal{B}(\mathcal{H}_c)\,,\ ||\psi(\hat{x}_1, \hat{x}_2)||_{HS} := \sqrt{\text{tr}_\text{c}(\psi^\dagger(\hat{x}_1, \hat{x}_2)\psi(\hat{x}_1, \hat{x}_2))} < \infty\right\}. \tag{2.23}$$

Here tr_c denotes tracing over $\mathcal{H}_c$ and $\mathcal{B}(\mathcal{H}_c)$ represents the set of bounded operators on $\mathcal{H}_c$. This space $\mathcal{H}_q$ has a natural inner product:- $(\phi(\hat{x}_1, \hat{x}_2), \psi(\hat{x}_1, \hat{x}_2)) = \text{tr}_\text{c}(\phi(\hat{x}_1, \hat{x}_2)^\dagger\psi(\hat{x}_1, \hat{x}_2)) < \infty$ and forms a Hilbert space on its own. This space is the analog of the Hilbert space of square integrable wave functions of commutative (standard) quantum mechanics. It is easy to recognise that one can identify $\mathcal{H}_q$ as $\mathcal{H}_c \otimes \mathcal{H}_c^*$, with $\mathcal{H}_c^*$ being the dual of $\mathcal{H}_c$ (Eq. (2.21)). As a convention, we use angular ($|. >$) and round ($|.)$) kets in order to distinguish between vectors belonging to $\mathcal{H}_c$ and $\mathcal{H}_q$ respectively. Moreover, we shall use capital letters to distinguish operators acting on quantum Hilbert space $\mathcal{H}_q$ from those acting on non-commutative configuration space $\mathcal{H}_c$. Finally, it is easy to check that an unitary representation of the noncommutative Heisenberg algebra (Eq. (2.20)) is obtained through the following irreducible actions of $\hat{X}_i$'s and $\hat{P}_i$'s on $\mathcal{H}_q$:

$$\hat{X}_i\psi(\hat{x}_1, \hat{x}_2) = \hat{x}_i\psi(\hat{x}_1, \hat{x}_2) \ , \ \hat{P}_i\psi(\hat{x}_1, \hat{x}_2) = \frac{\hbar}{\theta}\epsilon_{ij}[\hat{x}_j, \psi(\hat{x}_1, \hat{x}_2)] \tag{2.24}$$

namely, the position acts by left multiplication and the momentum adjointly. It will be useful to introduce the following operators at this stage -

$$\begin{aligned} B &= \frac{1}{\sqrt{2\theta}}\left(\hat{X}_1 + i\hat{X}_2\right), \quad B^{\ddagger} = \frac{1}{\sqrt{2\theta}}\left(\hat{X}_1 - i\hat{X}_2\right), \\ \hat{P} &= \hat{P}_1 + i\hat{P}_2, \quad \hat{P}^{\ddagger} = \hat{P}_1 - i\hat{P}_2. \end{aligned} \tag{2.25}$$

where the notation $\ddagger$ has been used for Hermitian conjugation on $\mathcal{H}_q$ as opposed to $\dagger$ on $\mathcal{H}_c$. They act in the following way,

$$\begin{aligned} B\psi(\hat{x}_1, \hat{x}_2) &= b\psi(\hat{x}_1, \hat{x}_2), \\ B^{\ddagger}\psi(\hat{x}_1, \hat{x}_2) &= b^{\dagger}\psi(\hat{x}_1, \hat{x}_2), \\ P\psi(\hat{x}_1, \hat{x}_2) &= -i\hbar\sqrt{\frac{2}{\theta}}[b, \psi(\hat{x}_1, \hat{x}_2)], \\ P^{\ddagger}\psi(\hat{x}_1, \hat{x}_2) &= i\hbar\sqrt{\frac{2}{\theta}}[b^{\dagger}, \psi(\hat{x}_1, \hat{x}_2)]. \end{aligned} \tag{2.26}$$

As the position coordinates $\hat{x}_1$ and $\hat{x}_2$ no longer commute here, common eigenstates of these operators do not exist. In such a situation, the best one can do is to introduce the minimal uncertainty states i.e. maximally localized coherent states on $\mathcal{H}_c$ as-

$$|z\rangle = e^{-z\bar{z}/2}e^{zb^\dagger}|0\rangle \;\in\; \mathcal{H}_c \quad ; \quad \Delta\hat{x}_1\Delta\hat{x}_2 = \frac{1}{2}\theta \tag{2.27}$$

where, $z = \frac{1}{\sqrt{2\theta}}(x_1 + ix_2)$ is a dimensionless complex number. These coherent states furnish an overcomplete basis on the non-commutative configuration space $\mathcal{H}_c$. Corresponding to these states we can construct a state (HS operator) in $\mathcal{H}_q$ as follows-

$$|z,\bar{z}) \equiv |x_1, x_2) = \frac{1}{\sqrt{2\pi\theta}}|z\rangle\langle z| \in \mathcal{H}_q \quad ; \quad (z', \bar{z}'|z, \bar{z}) = e^{-|z-z'|^2} \tag{2.28}$$

One can easily verify that this is indeed a Hilbert-Schmidt operator by writing the trace in terms of coherent states and using $|\langle z|w\rangle|^2 = e^{-|z-w|^2}$. Also note that these states have the property, $B|z,\bar{z}) = z|z,\bar{z})$, which leads to the natural interpretation of (x_1, x_2) as the dimensionful position coordinates. Now, the 'position' representation of a state $|\psi)$ can be constructed by taking overlap with (Eq. (2.28)) as,

$$\psi(x_1, x_2) \equiv (z, \bar{z}|\psi) = \frac{1}{\sqrt{2\pi\theta}} tr_c(|z\rangle\langle z|\psi(\hat{x}_1, \hat{x}_2)) = \frac{1}{\sqrt{2\pi\theta}}\langle z|\psi(\hat{x}_1, \hat{x}_2)|z\rangle \tag{2.29}$$

The completeness relations for eigenstates $|z,\bar{z})$ of $\hat{B}$, which will be an important ingredient in the construction of path integral, reads:

$$\int 2\theta dz d\bar{z}\; |z, \bar{z}) \star (z, \bar{z}| = \int dx_1 dx_2\; |x_1, x_2) \star (x_1, x_2| = 1_Q \tag{2.30}$$

where the Voros star product between two functions $f(z,\bar{z})$ and $g(z,\bar{z})$ is defined as

$$f(z,\bar{z}) \star g(z,\bar{z}) = f(z,\bar{z}) e^{\overleftarrow{\partial_{\bar{z}}}\overrightarrow{\partial_z}} g(z,\bar{z}) \tag{2.31}$$

and is mathematically equivalent to the widely-used Moyal star-product in NC quantum mechanics which is given as follows:

$$f(x_1,x_2)\star_M g(x_1,x_2) = f(x_1,x_2)e^{\frac{i\theta}{2}\epsilon_{ij}\overleftarrow{\partial_i}\overrightarrow{\partial_j}}g(x_1,x_2)\ ; \quad i,j=x_1,x_2. \tag{2.32}$$

The two star products, on using $z = \frac{1}{\sqrt{2\theta}}(x_1+ix_2)$, can be related as, $\star = e^{\frac{\theta}{2}\overleftarrow{\partial_i}\overrightarrow{\partial_i}}\star_M$. The formulation to be used in the thesis is based on Hilbert-Schmidt operators and as discussed before, this involves the Voros star product only which will be denoted throughout the thesis with a $\star$ symbol as in (Eq. (2.31)). We, therefore, are not going to employ the Moyal star product ($\star_M$) in this thesis.

In this operator formulation of NCQM, a consistent position probability interpretation can indeed be provided by assigning the probability of finding the particle at position (x_1,x_2) through the notion of positive operator valued measure (POVM) as has been shown in [13]. Finally, we state that probability conservation also holds here and the norm of the state is preserved under time evolution just as in the commutative case, $\frac{\partial}{\partial t}(\psi|\psi) = 0$.

We now introduce the momentum eigenstates as-

$$|p) = \sqrt{\frac{\theta}{2\pi\hbar^2}}e^{\frac{i}{\hbar}\vec{p}.\hat{\vec{x}}} = \sqrt{\frac{\theta}{2\pi\hbar^2}}e^{i\sqrt{\frac{\theta}{2\hbar^2}}(\bar{p}b+pb^\dagger)}\ ;\ \hat{P}_i|p) = p_i|p)\ ;\ \ p = p_1+ip_2 \tag{2.33}$$

satisfying the usual completeness and orthonormality relations-

$$\int d^2p\ |p)(p| = 1_Q\ .\ ;\ \ (\vec{p'}|\vec{p}) = \delta^2(\vec{p'}-\vec{p}) \tag{2.34}$$

Therefore, the wave-function of a "free particle" on the noncommutative plane is given by

$$(z,\bar{z}|p) = \frac{1}{\sqrt{2\pi\hbar^2}}e^{-\frac{\theta}{4\hbar^2}\bar{p}p}e^{i\sqrt{\frac{\theta}{2\hbar^2}}(p\bar{z}+\bar{p}z)}\ . \tag{2.35}$$

On the other hand, one can also study deformed systems by mapping the NC coordinates (X_i) to commutative coordinates (X_i^c) through Bopp transformations [41, 42] given by:

$$X_i = X_i^c - \frac{\theta}{2\hbar}\epsilon_{ij}P_j \tag{2.36}$$

where,

$$[X_i^c, X_j^c] = 0; \quad [P_i, P_j] = 0; \quad [X_i^c, P_j] = i\hbar\delta_{ij} \tag{2.37}$$

This will be our approach for carrying out various analysis predominantly in subsequent chapters of the thesis where we will discuss more on this method and in section 2.3 of the present chapter, we will employ (Eq. (2.36)) to carry out canonical quantization.

Now with the formalism being at our disposal, we move forward to study dissipative quantum dynamics in quantum spaces, taken here to be the Moyal plane.

2.2 Path integral quantization

The formalism discussed in the previous section will be now utilised to write down the path integral for the propagation kernel on the two dimensional noncommutative plane, which has been first developed in [60] for the case of a free particle and the 2-D harmonic oscillator in Moyal plane. By making repeated use of resolution of identity relations (Eq. (2.30)) for each intermediate points which give rise to the infinitesimally small intervals (z_{n-1}, t_{n-1} to z_n, t_n), the finite time propagation kernel can be expressed in the noncommutative setting as,

$$(z_f, t_f|z_0, t_0) \quad = \quad \lim_{n\to\infty}\int\prod_{j=1}^{n}(dz_j d\bar{z}_j)\ (z_f, t_f|z_n, t_n) \star_n (z_n, t_n|....|z_1, t_1) \star_1 (z_1, t_1|z_0, t_0)\ . \tag{2.38}$$

The Hamiltonian (Eq. (2.17)) acting on the quantum Hilbert space $\mathcal{H}_q$ now reads

$$\hat{H} = \frac{\hat{\vec{P}}^2}{2\mu} + \frac{\gamma}{2\mu}(\hat{X}_1\hat{P}_2 - \hat{X}_2\hat{P}_1) + \frac{1}{2}\mu(\omega_1^2\hat{X}_1^2 + \omega_2^2\hat{X}_2^2) \tag{2.39}$$

Considering this form of the system Hamiltonian, we now move ahead to compute the transition matrix element over a small segment in the above path integral (Eq. (2.38)). With the help of equations (Eq. (2.34)) and (Eq. (2.35)), we have

$$\begin{aligned}(z_{j+1},t_{j+1}|z_j,t_j) &= (z_{j+1}|e^{-\frac{i}{\hbar}\epsilon\hat{H}}|z_j)\\ &= \int_{-\infty}^{+\infty} d^2p_j\ e^{-\frac{\theta}{2\hbar^2}\bar{p}_jp_j}e^{i\sqrt{\frac{\theta}{2\hbar^2}}[p_j(\bar{z}_{j+1}-\bar{z}_j)+\bar{p}_j(z_{j+1}-z_j)]}\\ &\quad\times e^{-\frac{i}{\hbar}\epsilon[\frac{\bar{p}_jp_j}{2\mu}+\frac{\mu\theta}{4}(\omega_1^2-\omega_2^2)(\bar{z}_{j+1}^2+z_j^2)+\frac{\mu\theta}{4}(\omega_1^2+\omega_2^2)(2\bar{z}_{j+1}z_j+1)-\frac{i\gamma}{2\mu}\sqrt{\frac{\theta}{2}}(p_j\bar{z}_{j+1}-\bar{p}_jz_j)]}+O(\epsilon^2).\end{aligned}$$

On substituting the above expression in (Eq. (2.38)), we carry out the star products explicitly to obtain,

$$\begin{aligned}(z_f,t_f|z_0,t_0) = \quad & \lim_{n\to\infty}\int\prod_{j=1}^{n}(dz_jd\bar{z}_j)\prod_{j=0}^{n}d^2p_j\\ & \exp\left(\sum_{j=0}^{n}\left[\frac{i}{\hbar}\sqrt{\frac{\theta}{2}}\left[p_j\left\{\left(1+\frac{i\epsilon\gamma}{2\mu}\right)\bar{z}_{j+1}-\bar{z}_j\right\}+\bar{p}_j\left\{z_{j+1}-\left(1+\frac{i\epsilon\gamma}{2\mu}\right)z_j\right\}\right]+\sigma p_j\bar{p}_j\right]\right.\\ & \left.\qquad+\frac{\theta}{2\hbar^2}\sum_{j=0}^{n-1}p_{j+1}\bar{p}_j-\frac{i}{\hbar}\epsilon V(\bar{z}_{j+1},z_j)\right) \qquad (2.40)\end{aligned}$$

where

$$\sigma = -\left(\frac{i\epsilon}{2\mu\hbar}+\frac{\theta}{2\hbar^2}\right)$$

and,

$$V(\bar{z}_{j+1},z_j) = \frac{\mu\theta}{4}(\omega_1^2-\omega_2^2)(\bar{z}_{j+1}^2+z_j^2)+\frac{\mu\theta}{4}(\omega_1^2+\omega_2^2)(2\bar{z}_{j+1}z_j+1) \qquad (2.41)$$

The momentum integral can be performed easily since it has been brought to the Gaussian form $\exp\left(\sum_{i,j}p_iM_{i,j}\bar{p}_j\right)$, where M is a $N\times N$ ($N=n+1=T/\epsilon$, $T=t_f-t_0$) dimensional matrix given by

$$M_{lr} = \sigma\delta_{l,r} + \frac{\theta}{2\hbar^2}\delta_{l+1,r}\ . \tag{2.42}$$

Now one should take note of the fact that the appearance of off-diagonal terms here is a purely noncommutative effect. Furthermore, this matrix M is a special kind of Toeplitz matrix, generally known as the circulant matrix whose eigenvalues and normalised eigenvectors are given by [61] :

$$\begin{aligned}\lambda_k &= \sigma + \frac{\theta}{2\hbar^2}e^{2\pi ik/N} \quad ; \quad k \in [0, n] \\ u_k &= \frac{1}{\sqrt{N}}(1 \quad e^{2\pi ik/N} \quad e^{4\pi ik/N}....)^T\ . \end{aligned} \tag{2.43}$$

On executing the momentum integral, we obtain

$$\begin{aligned}(z_f, t_f|z_0, t_0) = \quad & \lim_{n\to\infty} A\int\prod_{j=1}^{n}(dz_j d\bar{z}_j)\exp\left(-\vec{\partial}_{z_f}\vec{\partial}_{\bar{z}_0}\right) \\ & \times\exp\left(\frac{\theta}{2\hbar^2}\sum_{l=0}^{n}\sum_{r=0}^{n}\left\{\left(1+\frac{i\epsilon\gamma}{2\mu}\right)\bar{z}_{l+1}-\bar{z}_l\right\}M^{-1}_{lr}\left\{z_{r+1}-\left(1+\frac{i\epsilon\gamma}{2\mu}\right)z_r\right\}\right) \\ & \times\exp\left(-\frac{i}{\hbar}\epsilon\sum_{j=0}^{n}V(\bar{z}_{j+1}, z_j)\right)\end{aligned} \tag{2.44}$$

where A is an unimportant constant for our purposes obtained from momentum integrations. We now make use of the fact that $z_l = z(l\epsilon)$ and $z_{l+1} - z_l = \epsilon\dot{z}(l\epsilon) + O(\epsilon^2)$.

$$
\begin{aligned}
(z_f, t_f|z_0, t_0) = & \lim_{n\to\infty} A \int \prod_{j=1}^{n} (dz_j d\bar{z}_j) \exp\left(-\vec{\partial}_{z_f}\vec{\partial}_{\bar{z}_0}\right) \exp\left(\frac{\theta\epsilon}{2\hbar^2 T} \sum_{l,r,k=0}^{n} \epsilon \left[\dot{\bar{z}}(l\epsilon) + \frac{i\gamma}{2\mu}\bar{z}(l\epsilon)\right]\right. \\
& \times \left[\sigma + \frac{\theta}{2\hbar^2} e^{\epsilon\partial_{(l\epsilon)}}\right]^{-1} [e^{2\pi i(l-r)k\epsilon/T}] \times \epsilon \left.\left[\dot{z}(r\epsilon) - \frac{i\gamma}{2\mu} z(r\epsilon)\right]\right) \exp\left(-\frac{i}{\hbar}\epsilon \sum_{j=0}^{n} V(\bar{z}_j, z_j)\right) \\
= & \lim_{n\to\infty} A \int \prod_{j=1}^{n} (dz_j d\bar{z}_j) \exp\left(-\vec{\partial}_{z_f}\vec{\partial}_{\bar{z}_0}\right) \\
& \times \exp\left(\frac{\theta}{2\hbar^2 T} \sum_{l,r,k=0}^{n} \epsilon \left[\dot{\bar{z}}(l\epsilon) + \frac{i\gamma}{2\mu}\bar{z}(l\epsilon)\right] \left[-\frac{i}{2\mu\hbar} + \frac{\theta}{2\hbar^2}\partial_{(l\epsilon)} + O(\epsilon^2)\right]^{-1}\right. \\
& \qquad \times [e^{2\pi i(l-r)k\epsilon/T}]\epsilon \left.\left[\dot{z}(r\epsilon) - \frac{i\gamma}{2\mu} z(r\epsilon)\right]\right) \times \exp\left(-\frac{i}{\hbar}\epsilon \sum_{j=0}^{n} V(\bar{z}_j, z_j)\right)
\end{aligned}
\tag{2.45}
$$

On taking the limit $\epsilon \to 0$ and performing the sum over k, we finally have -

$$
(z_f, t_f|z_0, t_0) = A \exp\left(-\vec{\partial}_{z_f}\vec{\partial}_{\bar{z}_0}\right) \int_{z(t_0)=z_0}^{z(t_f)=z_f} \mathcal{D}z\mathcal{D}\bar{z} \exp\left(\frac{i}{\hbar}S\right) \tag{2.46}
$$

where the action S is given as follows :

$$
\begin{aligned}
S = \int_{t_0}^{t_f} dt \quad & \left[\frac{\theta}{2}\left\{\dot{\bar{z}}(t) + \frac{i\gamma}{2\mu}\bar{z}(t)\right\}\left(\frac{1}{2\mu} + \frac{i\theta}{2\hbar}\partial_t\right)^{-1}\left\{\dot{z}(t) - \frac{i\gamma}{2\mu}z(t)\right\} - \frac{\mu\theta}{2}(\omega_1^2 + \omega_2^2)\bar{z}(t)z(t)\right. \\
& \left. - \frac{\mu\theta}{4}(\omega_1^2 - \omega_2^2)(z^2(t) + \bar{z}^2(t))\right]
\end{aligned}
\tag{2.47}
$$

The equation of motion following from the above action is of the following form :

$$
\ddot{z}(t) - i\left\{\frac{\gamma}{\mu} - \mu\theta\frac{(\omega_1^2 + \omega_2^2)}{2}\right\}\dot{z}(t) + \left\{\frac{\omega_1^2 + \omega_2^2}{2} - \frac{\gamma^2}{4\mu^2}\right\}z(t) = -\frac{i\mu\theta}{2}(\omega_1^2 - \omega_2^2)\dot{\bar{z}}(t) - \frac{(\omega_1^2 - \omega_2^2)}{2}\bar{z}(t) \tag{2.48}
$$

Splitting into real and imaginary parts, we get

$$\ddot{x}_1 + \gamma_1 \dot{x}_2 + \left\{ \omega_1^2 - \frac{\gamma^2}{4\mu^2} \right\} x_1 = 0 \qquad ; \quad \gamma_1 = \frac{\gamma}{\mu} - \frac{\mu\theta}{\hbar}\omega_2^2. \tag{2.49}$$

and,

$$\ddot{x}_2 - \gamma_2 \dot{x}_1 + \left\{ \omega_2^2 - \frac{\gamma^2}{4\mu^2} \right\} x_2 = 0 \qquad ; \quad \gamma_2 = \frac{\gamma}{\mu} - \frac{\mu\theta}{\hbar}\omega_1^2. \tag{2.50}$$

In [40] and [60], effective path-integral equations of motion has been obtained for a two-dimensional simple harmonic oscillator on the noncommutative plane. This can be matched with (Eq. (2.49),Eq. (2.50)) by setting $\gamma = 0$. Also, we notice that the expectation values in the coherent state basis of the operator-valued Heisenberg's equations of motion derived from the Hamiltonian (Eq. (2.39)) are identical to equations (Eq. (2.49)) and (Eq. (2.50)) derived from the path integral formalism. Now, in order to reproduce the commutative and classical equations of motion corresponding to the Hamiltonian (Eq. (2.17)), the limits $\theta, \hbar \to 0$ are taken in such a manner that $\frac{\theta}{\hbar} \to 0$ and the commutative equations are given by -

$$\ddot{x}_1 + \frac{\gamma}{\mu}\dot{x}_2 + \left\{ \omega_1^2 - \frac{\gamma^2}{4\mu^2} \right\} x_1 = 0 \quad . \tag{2.51}$$

and,

$$\ddot{x}_2 - \frac{\gamma}{\mu}\dot{x}_1 + \left\{ \omega_2^2 - \frac{\gamma^2}{4\mu^2} \right\} x_2 = 0 \quad . \tag{2.52}$$

Due to the presence of the non-commutative parameter θ, the coupling factors γ_1, γ_2 (Eq. (2.49),Eq. (2.50)) have been modified. Moreover they are anisotropic ($\gamma_1 \neq \gamma_2$) and this is basically due to the anisotropic nature of the harmonic potentials in (Eq. (2.17)) itself.

The characteristic frequencies of this coupled system of equations (Eq. (2.49)) and (Eq. (2.50)) are obtained as:

$$\Omega_{\pm} = \sqrt{\frac{\nu_1^2 + \nu_2^2 + \gamma_1\gamma_2}{2} \pm \frac{1}{2}\sqrt{\gamma_1\gamma_2(2\nu_1^2 + 2\nu_2^2 + \gamma_1\gamma_2) + (\nu_1^2 - \nu_2^2)^2}} \tag{2.53}$$

where,

$$\nu_1^2 = \omega_1^2 - \frac{\gamma^2}{4\mu^2} \quad , \quad \nu_2^2 = \omega_2^2 - \frac{\gamma^2}{4\mu^2}. \tag{2.54}$$

A special case of this more general result applies for a commutative anisotropic oscillator in a magnetic field whose normal modes were obtained in [62] using canonical quantization.

2.3 Canonical quantization

In this section, we will carry out the canonical quantization of our dynamical system in an effective commutative approach and obtain the energy spectrum. In this approach, one maps the noncommutative problem embedded in the ambient Moyal plane to an equivalent commutative problem in the commuative plane and where the phase space variables now satisfy the usual Heisenberg algebra (Eq. (2.37)). This is done by making use of the commuting coordinates $\hat{X}_i^c$ (Eq. (2.36)), rather than $\hat{X}_i$ to recast the Hamiltonian (Eq. (2.39)) as

$$\hat{H} = \frac{\hat{P_1}^2}{2\mu_1} + \frac{\hat{P_2}^2}{2\mu_2} + \frac{1}{2}\mu(\omega_1^2 \hat{X}_1^{c\,2} + \omega_2^2 \hat{X}_2^{c\,2}) + \frac{\gamma_2}{2}\hat{X}_1^c\hat{P}_2 - \frac{\gamma_1}{2}\hat{X}_2^c\hat{P}_1 \tag{2.55}$$

where

$$\mu_1 = \frac{\mu}{(1 - \frac{\gamma\theta}{2\hbar} + \frac{\mu^2\theta^2\omega_2^2}{4\hbar^2})} \; ; \qquad \mu_2 = \frac{\mu}{(1 - \frac{\gamma\theta}{2\hbar} + \frac{\mu^2\theta^2\omega_1^2}{4\hbar^2})} \tag{2.56}$$

and γ_1 and γ_2 are as previously defined in (Eq. (2.49)) and (Eq. (2.50)) respectively. To diagonalise the Hamiltonian we introduce the following canonical transformation :

$$\begin{pmatrix} \hat{X}_1^c \\ \hat{X}_2^c \\ \hat{P}_1 \\ \hat{P}_2 \end{pmatrix} = \begin{pmatrix} acosu & 0 & 0 & \frac{1}{b}sinu \\ 0 & acosu & \frac{1}{b}sinu & 0 \\ 0 & -bsinu & \frac{1}{a}cosu & 0 \\ -bsinu & 0 & 0 & \frac{1}{a}cosu \end{pmatrix} \begin{pmatrix} \hat{q}_1 \\ \hat{q}_2 \\ \hat{\pi}_1 \\ \hat{\pi}_2 \end{pmatrix} \tag{2.57}$$

The strange nature of this matrix is to be noted, which essentially has a block-diagonal form, which involves a "rotation" (i.e. modulo a, b) in $(\hat{X}_1^c, \hat{P}_2)$ plane and in $(\hat{X}_2^c, \hat{P}_1)$ plane. In terms of these transformed variables, the Hamiltonian reduces to -

$$\hat{H} = \sigma_1^2\hat{\pi}_1^2 + \sigma_2^2\hat{\pi}_2^2 + k_1^2\hat{q}_1^2 + k_2^2\hat{q}_2^2 + \lambda_1\hat{q}_1\hat{\pi}_2 + \lambda_2\hat{q}_2\hat{\pi}_1 \ . \tag{2.58}$$

where,

$$\begin{aligned}
k_1^2 &= \frac{b^2}{2\mu_2}\sin^2 u + \frac{\mu\omega_1^2 a^2}{2}\cos^2 u + \frac{\gamma_1 ab}{2}\sin 2u \\
k_2^2 &= \frac{b^2}{2\mu_1}\sin^2 u + \frac{\mu\omega_2^2 a^2}{2}\cos^2 u - \frac{\gamma_2 ab}{2}\sin 2u \\
\sigma_1^2 &= \frac{1}{2\mu_1 a^2}\cos^2 u + \frac{\mu\omega_2^2}{2b^2}\sin^2 u + \frac{\gamma_2}{2ab}\sin 2u \\
\sigma_2^2 &= \frac{1}{2\mu_2 a^2}\cos^2 u + \frac{\mu\omega_1^2}{2b^2}\sin^2 u - \frac{\gamma_1}{2ab}\sin 2u \\
\lambda_1 &= -\frac{b}{2\mu_2 a}\sin 2u + \frac{m\omega_1^2 a}{2b}\sin 2u - \gamma_1\cos 2u \\
\lambda_2 &= -\frac{b}{2\mu_1 a}\sin 2u + \frac{m\omega_2^2 a}{2b}\sin 2u + \gamma_2\cos 2u
\end{aligned} \tag{2.59}$$

It's clear that if we set $\lambda_1 = 0 = \lambda_2$, the Hamiltonian will be diagonalized. The ratio of the parameters a, b and u in this situation are determined to be -

$$\frac{a}{b} = \sqrt{\frac{\mu_1\gamma_2 + \mu_2\gamma_1}{\mu\mu_1\mu_2(\gamma_2\omega_1^2 + \gamma_1\omega_2^2)}}$$

$$and, \quad \tan(2u) = \sqrt{\frac{4\mu_1\mu_2(\mu_1\gamma_2 + \mu_2\gamma_1)(\gamma_2\omega_1^2 + \gamma_1\omega_2^2)}{\mu(\mu_2\omega_1^2 - \mu_1\omega_2^2)^2}} \tag{2.60}$$

We now introduce the standard creation and annihilation operators as-

$$\hat{a}_1 = \sqrt{\frac{\sigma_1}{2\hbar k_1}}(\frac{k_1}{\sigma_1}\hat{q}_1 + i\hat{\pi}_1) \quad ; \quad \hat{a}_2 = \sqrt{\frac{\sigma_2}{2\hbar k_2}}(\frac{k_2}{\sigma_2}\hat{q}_2 + i\hat{\pi}_2) \tag{2.61}$$

satisfying $[\hat{a}_1, \hat{a}_1^\dagger] = 1 = [\hat{a}_2, \hat{a}_2^\dagger]$.

With this, we finally arrive at the diagonalised form of the Hamiltonian as :

$$\hat{H} = 2\hbar k_1\sigma_1(\hat{N}_1 + \frac{1}{2}) + 2\hbar k_2\sigma_2(\hat{N}_2 + \frac{1}{2}) \qquad ; \ \hat{N}_1 = \hat{a}_1^\dagger\hat{a}_1 \ , \ \hat{N}_2 = \hat{a}_2^\dagger\hat{a}_2 \ . \tag{2.62}$$

The energy eigenstates of the Hamiltonian can now be easily labelled by the integer eigenvalues of the number operators $\hat{N}_1, \hat{N}_2$ as $|n_1, n_2)$ satisfying

$$H|n_1, n_2) = \left(\hbar\tilde{\Omega}_1(n_1 + \frac{1}{2}) + \hbar\tilde{\Omega}_2(n_2 + \frac{1}{2})\right)|n_1, n_2) \tag{2.63}$$

with the characteristic frequencies $\tilde{\Omega}_1 = 2k_1\sigma_1$ and $\tilde{\Omega}_2 = 2k_2\sigma_2$. After inserting the values of $\sigma_1, \sigma_2, k_1, k_2$ from (Eq. (2.59)), we notice that these frequencies correspond exactly with the frequencies (Eq. (2.53)) derived from the effective equations of the two modes in the path-integral technique: $\tilde{\Omega}_1 = \Omega_+$ and $\tilde{\Omega}_2 = \Omega_-$.

2.4 Reduction to the Bateman oscillators:

Now in this section, we would like to investigate the feasibility of setting limiting values of the parameters ϵ, η introduced in (Eq. (2.11), Eq. (2.12)) that will result in Bateman oscillators. This cannot be implemented in any of the intermediate stages since the coordinates x_1', x_2' may fail to make any sense. Remarkably however, if we rewrite (Eq. (2.49), Eq. (2.50)) in terms of the original coordinates x, y (by successively using T_2^{-1} (Eq. (2.16)) and T_1^{-1} (Eq. (2.7))) while keeping the parameters ϵ, η, we get completely sensible expressions for (Eq. (2.49), Eq. (2.50)) - [2]

$$\ddot{x} + \eta\ddot{y} + (\gamma + \frac{\theta\omega^2}{\hbar} - \frac{\theta\gamma^2}{4\hbar} - \frac{\epsilon\eta\theta}{\hbar})\dot{x} + (\frac{\epsilon\theta}{\hbar} - \frac{\eta\theta\omega^2}{\hbar})\dot{y} + \epsilon y + \omega^2 x = 0 \quad . \tag{2.65}$$

and,

$$\ddot{y} + \eta\ddot{x} - (\gamma + \frac{\theta\omega^2}{\hbar} - \frac{\theta\gamma^2}{4\hbar} - \frac{\epsilon\eta\theta}{\hbar})\dot{y} - (\frac{\epsilon\theta}{\hbar} - \frac{\eta\theta\omega^2}{\hbar})\dot{x} + \epsilon x + \omega^2 y = 0 \quad , \tag{2.66}$$

in the sense that now one can take a smooth limit $\epsilon, \eta \to 0$ to obtain -

$$\ddot{x} + \gamma_R\dot{x} + \omega^2 x = 0 \tag{2.67}$$

and,

$$\ddot{y} - \gamma_R\dot{y} + \omega^2 y = 0 \quad ; \ \gamma_R = \gamma + \frac{\theta\omega^2}{\hbar} - \frac{\theta\gamma^2}{4\hbar}. \tag{2.68}$$

Thus, we have restored the form of Bateman oscillators (Eq. (2.1), Eq. (2.2)) except that the damping factor γ now takes a "renormalized" value γ_R. Both quantum and noncommutative effects are incorporated here.

[2] The limit $\eta \to 1$, which corresponds to a constraint system as follows from the structure of the Lagrangian (Eq. (2.13)) itself, resulting in the Dirac bracket -

$$[x, y]_{DB} = \frac{4\gamma}{7\gamma^2 + 8(\epsilon - \omega^2)} \tag{2.64}$$

following Dirac's constraint analysis [63], is not of relevance here.

Furthermore, now notice that the characteristic frequencies of the pure Bateman system can be obtained by taking the limits $\epsilon, \eta \to 0$ in (Eq. (2.53)). We then observe that the system becomes isotropic ($\omega_1^2 = \omega_2^2 = \omega^2 - \frac{\gamma^2}{4}$) and the mass parameter μ becomes purely imaginary: $\mu = i$. Not only that, the coupling factors γ_1 and γ_2 (Eq. (2.49), Eq. (2.50)) also become equal for the two modes. However, these are purely imaginary and we end up obtaining the characteristic frequencies as-

$$\lambda_{\pm}^{R} = \pm i\frac{\gamma_R}{2} + \sqrt{\omega^2 - \frac{\gamma_R^2}{4}} \tag{2.69}$$

Surprisingly, the same characteristic frequencies may be extracted straight from (Eq. (2.67), Eq. (2.68)). These frequencies are also strikingly similar to $\lambda_{\pm}$, with the exception that γ is replaced with γ_R now. Again, the key critical ratio is $R = \frac{2\omega}{\gamma_R}$, which governs the dynamics in this case as well, and if it surpasses unity ($R > 1$), we get oscillatory, damped, and anti-damped solutions, which correspond to the two modes 'x' and 'y' of the pure Bateman system.

Moreover, it is interesting to notice at this point that the same frequencies $\lambda_{\pm}^{R}$ are also obtained by taking appropriate limits of the frequencies $\tilde{\Omega}_1, \tilde{\Omega}_2$ in (Eq. (2.63)). However, since the frequencies will turn out to be complex, it then becomes evident that one needs to restrict to the diagonal subspace if one has to retain reality of energy eigenvalues. In other words, the reality of the energy spectrum of the quantum Bateman system is ensured by a restriction on the allowed energy eigenstates to $\mathcal{H}_D := span\{|n,n)\} \subset span\{|n_1,n_2)\}$, by making it diagonal with respect to n_1 and n_2 modes. In such a case, the spectrum of the Hamiltonian turns out to be:

$$H|n,n) = 2\hbar \ \mathrm{Re}.(\tilde{\Omega}) \ (n + \frac{1}{2})|n,n) \tag{2.70}$$

where now after taking the relevant limits ($\epsilon, \eta \to 0$), we have $\tilde{\Omega}_1 = \tilde{\Omega}_2 = \tilde{\Omega}$. And finally,

$$H|n,n) = \hbar\Omega_B(n + \frac{1}{2})|n,n) \ \ ; \ \ \Omega_B = \sqrt{4\omega^2 - \gamma_R^2} \tag{2.71}$$

In the commutative limit ($\theta = 0$), this Ω_B fits the frequency found using 't Hooft's approach (eqn. (15) of [12]) and also by other methods in [64], [65]. The Bateman spectrum has a lower bound now and the above discussed restriction of the total Hilbert space $\mathcal{H}$ into $\mathcal{H}_D$ is a representative of the physicality condition of the Hilbert space appearing in t'Hooft's method for quantizing such a class of Hamiltonians.

Finally, we return back to the pair of equations (Eq. (2.67)) & (Eq. (2.68)), about which we make two crucial observations, which apply to two alternative scenarios:

.Case-I: Starting with $\gamma = 0$ initially, we encounter an effective γ_R which turns out to be non-zero: $\gamma_R = \frac{\theta\omega^2}{\hbar}$. This suggests that quantum effects in combination with noncommutativity, can cause damping even when the 'bare' damping parameter is absent.

.Case-II: Here $\gamma \neq 0$ and $\frac{\gamma^2}{4} > \omega^2$, i.e. $R < 1$. In this case, we can regard (Eq. (2.67), Eq. (2.68)) as a one parameter family of effective commutative pair of Bateman oscillators with 'running' damping constant γ_R (Eq. (2.68)) upon varying θ and the extremal point $\theta = 0$ yield the original Bateman pairs (1,2) in commutative plane $\mathbb{R}^2$. Further, we can fine-tune θ, taken as a free parameter now, such that $\gamma_R = 0$. This happens at the following value of θ: $\theta_c = \frac{\gamma\hbar}{\frac{\gamma^2}{4} - \omega^2}$. In other words, this indicates that an original dissipative theory in commutative space can be mapped to a non-dissipative NC theory i.e. the original pair of equations (Eq. (2.1), Eq. (2.2)) are mapped to the pair (Eq. (2.67), Eq. (2.68)) with $\gamma_R = 0$, when placed in the Moyal plane with NC parameter having the precise value θ_c. As a result, using a fine-tuned NC parameter θ_c, we may eliminate the damping or anti-damping solutions and preserve only the oscillatory component in a noncommutative space.

Therefore our observations demonstrate, in a certain sense, that we can trade between noncommutativity and dissipation establishing a 'duality' between these two aspects. In this regard, [66], [67] discuss other kinds of duality in the context of NC theories. In the end, we would like to point out that this link between noncommutativity and dissipation naturally corroborates, to some extent, 't Hooft's hypothesis [10] that the physics of fields near the Planck scale l_p is essentially governed by viscosity, as previously stated if $\theta \sim l_p^2 = \frac{\hbar G}{c^3}$. Indeed, in this situation, $\frac{\theta}{\hbar} \sim \frac{G}{c^3}$ and the generated dissipation in Case-I (see previous page) can be viewed as solely gravitational effects, and they may be connected to black holes and information loss. Thus, we have elaborated through this simple model that it is possible to get a hint towards a suggestive role of dissipation in Planck scale physics, establishing a duality between noncommutativity and information loss.

2.5 Remarks

In this chapter, by considering our "master equations" (Eq. (2.11)) and (Eq. (2.12)), we have essentially tackled two different problems at once, namely the Landau problem with anisotropic oscillator potentials and the Bateman oscillators - the reason we named them as "master equations". It is worth remembering that the Hamiltonian (Eq. (2.17)) is just that of a charged particle travelling in a plane subjected to a normal magnetic field and trapped in anisotropic harmonic oscillator potentials where the damping constant plays the role of magnetic field. We have determined the spectrum of this problem (Eq. (2.53)). The spectrum of Bateman oscillators was then obtained at an appropriate stage by taking vanishing limits of the parameters ϵ, η. It is worth noting that the final system is also a closed system, with each independent mode being a damped (or anti-damp) harmonic oscillator with the spectrum in the commutative limit matching those existing in the literature. Our theoretical findings could have implications in optical microresonators and quantum decoherence phenomenon [59].

The present analysis provides a new direction in studying dissipative systems as compared with earlier works, which we feel, is aligned also in the spirit of 't Hooft's scheme of quantization. Further, our quantization was carried out completely by using HS operator formulation in both path integral and canonical quantization schemes. We found that NC can lead to dissipation, which may shed some light on the physics of fields at the Planck scale where the effects of noncommutativity is expected to most likely arise.

Chapter 3

Entropy in deformed dynamics

In the previous chapter, we have seen how noncommutativity allows for dissipative dynamics in an otherwise conserved system. Dissipative dynamics are essentially irreversible and are associated with entropy production. We have demonstrated that a sort of duality exists between dissipation and noncommuativity in the previous chapter. This gives an indication that noncommuative deformed systems also may lead to entropy production. Here, in this chapter we therefore want to investigate entropic aspects of noncommutative (deformed) systems. Due to its universality in different physical systems, the concept of entropy has become a central theme in modern physics. Various new relations have emerged between quantum theories and gravitational physics through study of entropic aspects [68]. Besides, thermodynamics and statistical mechanics, the closely associated concept of entanglement entropy has begun to play a central role in diverse areas of physics from condensed matter systems to gravitation and quantum field theory [69–73] and plays an important role in the current understanding of black hole physics [74, 75]. Entropy is a ubiquitous concept in quantum information theory.

Furthermore, in planar quantum mechanics, noncommutative algebra has been obtained through an enlarged phase space formulation in connection with t' Hooft's conjecture of dissipation and quantisation [35]. Hinted by the above considerations, we are motivated to explore the emergence of entanglement entropy in deformed quantum systems such as the Moyal plane and the noncommuative phase space (to be introduced in section 3.2 of this chapter). We follow here a Fock-space operator approach, unlike in [74, 75] where wave-functions were used to compute entropy for coupled 2D oscillators. We find a non-zero entropy which depends on noncommutative parameter θ and vanishes in the commutative limit. Here in this context, we will demonstrate that the physical model explicitly implementing this setup is a very generic problem in condensed matter in which one considers the non-relativistic motion of a charged particle on a plane with a transverse applied external magnetic field and a harmonic potential. This is the so-called "generalized Landau problem" (i.e. Landau problem in the presence of a 2D harmonic potential).

It is also worthful to remind here that the noncommutative parameter θ can be identified with the nonrelativistic anyonic spin parameter as discussed previously. The effect of exotic statistics between the anyons is given by a long-range gauge interaction i.e. a magnetic point-vortex (known as a statistical gauge potential in this context) and the ground-state of anyons in a harmonic potential can be explicitly computed [76]. Now, in 2+1 dimensions, apart from this singular magnetic flux, we also have Landau magnetic potentials. In this chapter, we show a connection between exotic oscillators and the Landau system in presence of an additional 2-D harmonic potential. By using this correspondence, we demonstrate that there exists an analog of the Unruh effect in the "generalised Landau problem".

We also extend our results to the case of noncommutative phase space, which is a more general setting of NC where, momentum components are also noncommutative along with the noncommutative position coordinates. Indeed, from Born's "principle of reciprocity" [14], which is motivated by the occurrence of equivalent dual descriptions, namely coordinate and momentum representations in quantum mechanics, it can be argued that noncommutative phase space is one of the most natural approaches to combine quantum mechanics with general relativity [9, 15, 77]. We shall use von-Neumann's notion of entropy as related to the reduced density matrix, called the entanglement entropy, to demonstrate the entanglement physics of a two-dimensional harmonic oscillator in the deformed Moyal plane ("exotic oscillator") and also in the noncommutative phase space. This chapter has been organised as follows: a) Section 3.1 deals with noncommutative configuration space where a naive approach is taken to show the emergence of entanglement entropy and b) In section 3.2, phase-space noncommutativity is considered and the entanglement entropy is computed here in a more formal and rigorous manner which is more transparent and brings out certain intriguing mathematical features of these two-dimensional oscillators in deformed systems where the entropy will now be shown to depend on a pair of noncommutative parameters. It is pertinent to mention here that all the equations derived in this section will have a smooth limit to recover results of section 3.1. Also, a discussion of squeezing effect in quantum optics will be made in relation to the present computational aspect. Finally in section 3.3, we will discuss a variant of Landau problem (generalized Landau problem) having a one-to-one correspondence with the exotic oscillator system and an Unruh-like effect will then be uncovered for the above problem.

3.1 Exotic oscillator system

We have seen in the previous chapter that the phase space coordinates of a nonrelativistic particle moving in the noncommutative Moyal plane satisfy the following four-dimensional NC Heisenberg algebra given by (Eq. (2.20)). We have already seen in the previous chapter how a consistent formulation of such a NCQM can be done using HS operators [13, 39, 60], however in the present chapter, we make use of a different approach and shall be working with an effective commutative theory. This is essentially a phase-space approach and it involves solving the deformed algebra (Eq. (2.20)) in terms of canonical (commutative) variables rather than using the noncanonical (noncommutative) variables in the theory [41, 42, 78], which can always be done as is guaranteed by Darboux's theorem which asserts that any noncanonical symplectic form can always be reduced to the standard canonical form by means of linear transformations [79]. In this particular context it is the so-called Bopp transformations which relates noncommutative variables with commutative variables. This procedure allows one to proceed with the standard canonical quantization of the theory as is customary with Faddeev-Jackiw quantization scheme [43, 80].

To that end, first observe that the above algebra can be realized in terms of commutative position coordinates q_i by the well-known Bopp transformations given as,

$$\hat{x}_i = q_i - \frac{\theta}{2\hbar}\epsilon_{ij}\pi_j \ ; \quad \hat{p}_i = \pi_i \tag{3.1}$$

where the associated phase space variables (q_i, π_j) satisfy the standard "commutative" Heisenberg algebra:

$$[q_i, q_j] = 0 \ ; \quad [\pi_i, \pi_j] = 0 \ ; \quad [q_i, \pi_j] = i\hbar\delta_{ij}. \tag{3.2}$$

Note that we have used over-head hat to denote noncommutative phase space operators, in contrast to their commutative counterparts, which are denoted without the hat notation.

The Hamiltonian of a 2-D NC harmonic oscillator embedded in the Moyal plane is given by-

$$H = \frac{1}{2m}(\hat{p}_1^2 + \hat{p}_2^2) + \frac{1}{2}m\omega^2(\hat{x}_1^2 + \hat{x}_2^2) \tag{3.3}$$

The Hamilton's equations of motion turn out to be,

$$\ddot{\hat{x}}_i - \frac{m\omega^2\theta}{\hbar}\epsilon_{ij}\dot{\hat{x}}_j + \omega^2\hat{x}_i = 0 \tag{3.4}$$

Now as mentioned earlier, for this analysis we will follow the effective commutative description. The effective commutative Hamiltonian is obtained by using (Eq. (3.1)) in (Eq. (3.3)) to get -

$$H = \frac{1}{2m_R}(\pi_1^2 + \pi_2^2) + \frac{1}{2}m_R\omega_R^2(q_1^2 + q_2^2) - \frac{m_R\omega_R^2\theta}{2\hbar}(q_1\pi_2 - q_2\pi_1) \tag{3.5}$$

Note that the effect of the deformation in the symplectic structure has been traded for as a modification in the form of the Hamiltonian which now contains a Zeeman-like interaction in the last term. Also notice that the mass and the oscillator frequencies are modified as,

$$\frac{1}{m_R} = \frac{1}{m} + \frac{m\omega^2\theta^2}{4\hbar^2} = \frac{\kappa^2}{m} \;\; ; \; \omega_R = \omega\kappa \tag{3.6}$$

and the dimensionless parameter κ is a very important one in the present context, as it essentially contains all the information of deformation and is given by,

$$\kappa = \sqrt{1 + \frac{m^2\omega^2\theta^2}{4\hbar^2}} \tag{3.7}$$

Next, we need to figure out the characteristic modes of the noncommutative system. To get the normal modes, we work with this effective commutative Hamiltonian and start with the standard sets of $\theta-$independent creation and annihilation operators -

$$b_i = \sqrt{\frac{m\omega}{2\hbar}}(q_i + i\frac{1}{m\omega}\pi_i); \;\; i = 1, 2 \tag{3.8}$$

obeying $[b_i, b_j^\dagger] = \delta_{ij}$. Note that the Hamiltonian (Eq. (3.5)) will correspond to a planar isotropic oscillator in the commutative ($\theta \to 0$) limit and is diagonalized by these creation and annihilation operators (b_i and $b_i^\dagger$). However, because of the presence of additional $\theta-$dependent terms in the Hamiltonian, for diagonalization we need to introduce suitably deformed creation and annihilation operators ($a_+(\theta), a_-(\theta)$) to be described below satisfying isomorphic commutation relations. In fact, we will now introduce such types of deformed operators to re-write the Hamiltonian (Eq. (3.5)) as,

$$H = \hbar\omega\kappa\left(a_i^\dagger(\theta)a_i(\theta) + 1\right) + i\hbar\gamma\left(a_1^\dagger(\theta)a_2(\theta) - a_1(\theta)a_2^\dagger(\theta)\right); \; \gamma = \frac{m\omega^2\theta}{2\hbar} \tag{3.9}$$

where these new deformed operators are given by:

$$a_i(\theta) = \sqrt{\frac{m\omega}{2\hbar\kappa}}(q_i + i\frac{\kappa}{m\omega}\pi_i) \quad ; \; i = 1, 2 \tag{3.10}$$

fulfilling

$$[a_i(\theta), a_j^\dagger(\theta)] = \delta_{ij} \quad ; \quad [a_i(\theta), a_j(\theta)] = 0 \tag{3.11}$$

Also note that in the $\theta = 0$ limit, κ becomes unity and naturally we have,

$$a_i(\theta)|_{\theta=0} = b_i \tag{3.12}$$

The Hamiltonian (Eq. (3.5)) then turns out to be -

$$H = \hbar\omega\kappa\Big(a_i^\dagger(\theta)a_i(\theta) + 1\Big) + i\hbar\gamma\Big(a_1^\dagger(\theta)a_2(\theta) - a_1(\theta)a_2^\dagger(\theta)\Big); \; \gamma = \frac{m\omega^2\theta}{2\hbar} \tag{3.13}$$

The Hamiltonian is finally diagonalized by implementing a $U(2)$ transformation as follows:

$$\begin{pmatrix} a_1(\theta) \\ a_2(\theta) \end{pmatrix} \longrightarrow \begin{pmatrix} a_+(\theta) \\ a_-(\theta) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} a_1(\theta) \\ a_2(\theta) \end{pmatrix} \tag{3.14}$$

fulfilling

$$[a_\pm(\theta), a_\pm^\dagger(\theta)] = 1 \; , [a_+(\theta), a_-(\theta)] = 0 = [a_+^\dagger(\theta), a_-^\dagger(\theta)] \tag{3.15}$$

so that the Hamiltonian (Eq. (3.13)) when re-casted in terms of $a_\pm(\theta)$ and $a_\pm^\dagger(\theta)$, indeed gets diagonalized as:

$$H = \hbar\omega_+\Big(a_+^\dagger(\theta)a_+(\theta) + \frac{1}{2}\Big) + \hbar\omega_-\Big(a_-^\dagger(\theta)a_-(\theta) + \frac{1}{2}\Big); \; \omega_\pm = \omega\kappa \mp \gamma \tag{3.16}$$

where we can easily identify $\omega_\pm$ as the characteristic frequencies.

The ground state of the system can therefore be readily expressed as the tensor product of individual ground states as-

$$|0,0;\theta>= \; |0;\theta>_+ \otimes |0;\theta>_- \quad ; \; a_\pm(\theta)|0;\theta>_\pm = 0 \tag{3.17}$$

and the non-degenerate energy eigen-states are given by

$$|n_+, n_-;\theta\rangle = \frac{\left(a_+^\dagger(\theta)\right)^{n_+}\left(a_-^\dagger(\theta)\right)^{n_-}}{\sqrt{n_+!}\sqrt{n_-!}}\,|0,0;\theta\rangle \tag{3.18}$$

such that the ground state has a separable form at this stage. However, as we shall show now that the same state $|0,0;\theta>$ can also be re-expressed in terms of harmonic oscillator states of a pair of ordinary isotropic oscillators in a commutative plane, which necessarily involves all the excited states as well. The appearance of entropy in certain bipartite systems when computed in a particular basis is a common feature of entropy calculation and is guided by physical narratives, see for example the discussion in [75]. Quite judiciously, our choice of decomposing the total Hilbert space into a direct product of two Hilbert spaces representing independent commutative oscillators is the most natural and instinctive one in the present context and will be more transparent in section 3.3.

Now the respective deformed raising/lowering operators $(a_i(\theta), a_i^\dagger(\theta))$ and their commutative counterparts $(b_i, b_i^\dagger)$ are now found to be related by

$$a_i(\theta) = \frac{1}{2}\left[Ab_i + Bb_i^\dagger\right] \tag{3.19}$$

where, $A = \frac{1}{\sqrt{\kappa}} + \sqrt{\kappa}, B = \frac{1}{\sqrt{\kappa}} - \sqrt{\kappa}$. This transformation also has a different form which will be made explicit in the next section.

Eventually, we can rewrite using (Eq. (3.14)) the normal mode creation and annihilation operators

($a_\pm$ and their hermitian conjugates) in terms of the commutative oscillators b_i's to get -

$$a_+(\theta) = \frac{1}{2}\left[Ab_+ + Bb_-^\dagger\right] \tag{3.20}$$

and,

$$a_-(\theta) = \frac{i}{2}\left[Ab_- + Bb_+^\dagger\right] \tag{3.21}$$

where, $b_+ = \frac{b_1 - ib_2}{\sqrt{2}}, b_- = \frac{b_1 + ib_2}{\sqrt{2}}$. We also have $a_\pm(\theta)|_{\theta=0} = b_\pm$.

The stage is now ready to expand the ground state $|0,0;\theta>$ in terms of the commutative number state basis $|n_\pm;\theta=0>_\pm$ as,

$$|0,0;\theta\rangle \quad = \quad \sum_{n_\pm=0}^{\infty} C_{n_+,n_-}(\theta)\ |n_+;\theta=0>_+ \otimes |n_-;\theta=0>_- \tag{3.22}$$

The coefficients $C_{n_+,n_-}(\theta)$ can be determined using the conditions:- $a_\pm(\theta)|0;\theta >_\pm = 0$ thus yielding the following two equations-

$$C_{n_+,n_-}(\theta) = -\sigma\sqrt{\frac{m}{n}}C_{n_+-1,n_--1}(\theta) \tag{3.23}$$

and,

$$C_{n_+,n_-}(\theta) = -\sigma\sqrt{\frac{n}{m}}C_{n_+-1,n_--1}(\theta) \tag{3.24}$$

where, $\sigma = \frac{B}{A}$ and, $-1 < \sigma < 0$. The two equations are simultaneously satisfied for -

$$C_{n_+,n_-}(\theta) = (-1)^n\sigma^n\delta_{n_+,n_-} \tag{3.25}$$

Therefore, the ground state is now determined as-

$$|0,0;\theta\rangle = \sum_{n=0}^{\infty}\ (-1)^n\sigma^n|n_+;\theta=0>_+ \otimes\ |n_-;\theta=0>_- \tag{3.26}$$

and the normalized ground states turn out to be-

$$|0,0;\theta\rangle = \sqrt{1-\sigma^2}\sum_{n=0}^{\infty}(-1)^n\sigma^n|n_+;\theta=0>_+\otimes|n_-;\theta=0>_- \tag{3.27}$$

Finally, the ground state density matrix is then given by:

$$\begin{aligned}\rho &= |0,0;\theta><0,0;\theta| \\ &= (1-\sigma^2)\sum_{n,m=0}^{\infty}(-\sigma)^{n+m}|n;\theta=0>_+<m;\theta=0|\otimes\ |n;\theta=0>_-<m;\theta=0|\end{aligned} \tag{3.28}$$

These are the desired expressions of the ground state $|0,0;\theta>$ and the corresponding density matrix in terms of states of the commutative planar oscillator states. And once re-expressed in this form, it ceases to be separable anymore. To show this, we focus our attention to subsystem '+' and carry out partial tracing over subsystem '−' to obtain the reduced density matrix for the subsystem '+' as,

$$\rho_+\equiv\rho_r = Tr_-\rho = (1-\sigma^2)\sum_{n=0}^{\infty}(\sigma^2)^n\ |n;\theta=0>_+<n;\theta=0| \tag{3.29}$$

Notice that this is a mixed state density matrix as $\rho_r^2\neq\rho_r$ unlike the total density matrix ρ in (Eq. (3.28)). It is now easy to identify the eigenvalues p_n of the reduced density matrix ρ_r which is given by:

$$p_n = (1-\sigma^2)\sigma^{2n} \tag{3.30}$$

Therefore, the entanglement entropy of the system is given by

$$\begin{aligned}S = &\quad -Tr.(\rho_r\log\rho_r) \\ = &\quad -\sum_{n=0}^{\infty}p_n\ \log p_n \\ = &\quad -\log\big(1-\sigma^2\big)-\frac{\sigma^2\log\sigma^2}{1-\sigma^2}\end{aligned} \tag{3.31}$$

where, $\sigma=\frac{B}{A}=\frac{1-\kappa}{1+\kappa}=\frac{1-\sqrt{1+\frac{m^2\omega^2\theta^2}{4\hbar^2}}}{1+\sqrt{1+\frac{m^2\omega^2\theta^2}{4\hbar^2}}}$

As a simple check, one can verify easily that in the commutative limit ($\theta \to 0$), the parameters $\kappa \to 1$ and $\sigma \to 0$ and thus the entanglement entropy vanishes. This signifies that the existence of a non-zero entropy in the system is purely due to the presence of noncommutativity between the two spatial degrees of freedom. Physically it means that if it is possible to probe the NC oscillator system using commutative oscillators/detectors, an intrinsic noise will always show up signifying the existence of this non-zero entropy (Eq. (3.31)).

It is imperative to mention here that the form of the reduced density matrix (Eq. (3.60)) resembles a thermal density matrix, describing an ensemble of one dimensional harmonic oscillators. This allows us to extract an effective temperature by recalling the form of a thermal mixed state density matrix for a one-dimensional harmonic oscillator of frequency ω -

$$\rho_{th} = \sum p_n \ |n; \theta = 0 >< n; \theta = 0| \tag{3.32}$$

where $p_n = (1 - e^{-\frac{\hbar\omega}{k_B T}})e^{-\frac{n\hbar\omega}{k_B T}}$

Now on comparing this with (Eq. (3.60)), we can readily identify the effective temperature T as,

$$\frac{1}{T} = -\frac{2k_B}{\hbar\omega} \log|\sigma| \tag{3.33}$$

Note that this temperature T (Eq. (3.33)) vanishes in the commutative limit, as it should be, since the reduced density matrix is then a pure state.

In the next section, we will show how our present ideas can be extended further to show the emergence of entropy in a more generalized noncommutative system where momentum components also do not commute apart from noncommuting position coordinates. Our analysis for the case of phase-space NC will be more sophisticated in the sense that it will bring out new relations between noncommutative and commutative theories. This will also pave the way for relating it qualitatively to the phenomenon of squeezing in a three-wave mixing process in a non-linear medium, as encountered in a typical quantum optical scenario. Finally we will discuss a variant of the well-known Landau problem and reveal up a Unruh-like effect in such systems by utilising a correspondence with the exotic oscillators.

3.2 Phase space noncommutativity

Now consider a more general and natural class of noncommutative structures arising in phase space giving rise to the following NC Heisenberg algebra :

$$[\hat{x}_i, \hat{x}_j] = i\theta_{ij} = i\theta\epsilon_{ij} \ ; \quad [\hat{p}_i, \hat{p}_j] = i\eta_{ij} = i\eta\epsilon_{ij} \ ; \quad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}. \tag{3.34}$$

Here η is the momentum noncommutative parameter and θ denotes the constant spatial noncommutative parameter that can be taken to be positive ($\theta > 0$) without loss of generality. In order to ensure consistent quantization following [81], we must have ($\theta\eta < 0$). This kind of noncommutative structure has been observed in certain crystals like $GaAs$, which exhibit Berry curvature in momentum space and in presence of an applied magnetic field on the crystal sample [16, 19, 82]. Here the corresponding noncommutativity parameters are related to the Berry curvature and the strength of applied magnetic field. Generally, momentum space NC arises in the presence of a background magnetic field as is the case in Landau problem, where the mechanical momentum components do not commute. The above phase space noncommutative structure is also believed to offer a more consistent approach for unifying quantum physics and general relativity at high energy scales [9, 14, 77] and can manifest itself in presence of curvature in both coordinate and momentum spaces [83].

The algebra (Eq. (3.34)) can be realized in terms of commutative position and momentum coordinates q_i, π_i by the following generalized Bopp tansformations [84] given as -

$$\hat{x}_i = q_i - \frac{\theta}{2\hbar}\epsilon_{ij}\pi_j + \lambda\epsilon_{ij}q_j \ ; \quad \hat{p}_i = \pi_i + \frac{\eta}{2\hbar}\epsilon_{ij}q_j + \lambda\epsilon_{ij}\pi_j \tag{3.35}$$

where the associated variables now satisfy the standard Heisenberg algebra (Eq. (3.2)) with $\lambda = \frac{\sqrt{-\theta\eta}}{2\hbar}$. In such a phase space, we now investigate a harmonic oscillator whose Hamiltonian is also of the same form as (Eq. (3.3)). However now the effective commutative Hamiltonian (Eq. (3.3)) in terms of the commutative coordinates and momenta q_i, π_i using the above generalized Bopp transformations (Eq. (3.35)) undergoes a further deformation as,

$$H = \alpha(\frac{{\pi_1}^2 + {\pi_2}^2}{2m}) + \beta\frac{1}{2}m\omega^2(q_1^2 + q_2^2) + \frac{\delta\omega}{2}(q_i\pi_i + \pi_i q_i) - \gamma(q_1\pi_2 - q_2\pi_1) \tag{3.36}$$

where,

$$\alpha = 1 + \lambda^2 + \frac{m^2\omega^2\theta^2}{4\hbar^2}; \quad \beta = 1 + \lambda^2 + \frac{\eta^2}{4m^2\omega^2\hbar^2}; \ \gamma = \frac{\eta}{2m\hbar} + \frac{m\omega^2\theta}{2\hbar}; \quad \delta = \lambda(\frac{\eta}{2m\omega\hbar} - \frac{m\omega\theta}{2\hbar}) \tag{3.37}$$

We now recall (Eq. (3.8)) the creation and annihilation operators for the commutative oscillators of frequency ω. The phase space operators q's and π's can be solved in terms of b and $b^\dagger$ as,

$$q_i = \sqrt{\frac{\hbar}{2m\omega}}\left(b_i + b_i^\dagger\right) \ ; \ \pi_i = -i\sqrt{\frac{\hbar m\omega}{2}}\left(b_i - b_i^\dagger\right) \tag{3.38}$$

Re-expressing the Hamiltonian (Eq. (3.36)) using above to get-

$$\begin{aligned} H = \quad & (\frac{2\beta}{\omega} + 2\alpha\omega)(\frac{b_i^\dagger b_i + b_i b_i^\dagger}{4}) + (\frac{\beta}{\omega} - \alpha\omega + 2i\delta)\frac{b_i^{\dagger 2}}{2} + (\frac{\beta}{\omega} - \alpha\omega - 2i\delta)\frac{b_i^2}{2} \\ & + i\hbar\gamma(b_1^\dagger b_2 - b_1 b_2^\dagger) \end{aligned} \tag{3.39}$$

Let us now invoke the following canonical transformation -

$$a_i(\theta,\eta) = \frac{1}{2}\left(\sqrt{\frac{\alpha}{\kappa'}} + \sqrt{\frac{\kappa'}{\alpha}} + i\frac{\delta}{\sqrt{\alpha\kappa'}}\right)b_i - \frac{1}{2}\left(\sqrt{\frac{\alpha}{\kappa'}} - \sqrt{\frac{\kappa'}{\alpha}} - i\frac{\delta}{\sqrt{\alpha\kappa'}}\right)b_i^\dagger \tag{3.40}$$

with,

$$\kappa' = \sqrt{\alpha\beta - \delta^2} = \sqrt{1 + 2\lambda^2 + \frac{m^2\omega^2\theta^2}{4\hbar^2} + \frac{\eta^2}{4m^2\omega^2\hbar^2}} \tag{3.41}$$

and by construction, $[a_i(\theta,\eta), a_j^\dagger(\theta,\eta)] = \delta_{ij}$. Note here that this transformation involves complex coefficients. However, with the use of appropriate parametrization, this may be expressed in a more illuminating manner as,

$$a_i(\theta,\eta) = \cosh\phi e^{i\psi} b_i - \sinh\phi e^{-i\xi} b_i^\dagger \tag{3.42}$$

where,

$$\tanh\phi = \sqrt{\frac{(\alpha - \kappa')^2 + \delta^2}{(\alpha + \kappa')^2 + \delta^2}}$$

$$\tan\psi = \frac{\delta}{\alpha + \kappa'} \quad ; \quad \tan\xi = \frac{\delta}{\alpha - \kappa'} \quad , \tag{3.43}$$

It is to be pointed here that in the $\eta = 0$ limit, this transformation (Eq. (3.42)) becomes the standard real Bogolyubov transformation and is identical to (Eq. (3.19)). Furthermore, it turns out that this transformation can also be implemented as a unitary transformation in the Hilbert space of states in a conventional manner as,

$$a_i(\theta,\eta) = U b_i U^\dagger \tag{3.44}$$

where the unitary operator is given by, $U = e^{-\frac{1}{2}(s b_j^2 - s^* b_j^{\dagger 2})}$. This is the single-mode squeezing transformation U with the complex squeezing parameter s given by, $s = \phi e^{-i(\psi+\xi)}$, generating non-classicality in the individual modes. The Hamiltonian H (Eq. (3.39)) turns out to be, in terms of these new (deformed) operators:

$$H = \hbar\omega\kappa'\Big(a_i^\dagger(\theta,\eta)a_i(\theta,\eta) + 1\Big) + i\hbar\gamma\Big(a_1^\dagger(\theta,\eta)a_2(\theta,\eta) - a_1(\theta,\eta)a_2^\dagger(\theta,\eta)\Big) \tag{3.45}$$

At the final stage, we carry out an additional $U(2)$ transformation on the following pair of annihilation operators -

$$\begin{pmatrix} a_1(\theta,\eta) \\ a_2(\theta,\eta) \end{pmatrix} \longrightarrow \begin{pmatrix} a_+(\theta,\eta) \\ a_-(\theta,\eta) \end{pmatrix} = \mathbf{M} \begin{pmatrix} a_1(\theta,\eta) \\ a_2(\theta,\eta) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} a_1(\theta,\eta) \\ a_2(\theta,\eta) \end{pmatrix} \tag{3.46}$$

where $\mathbf{M} \in U(2)$. These operators are related to the undeformed operators b_i using (Eq. (3.44)) as,

$$\begin{aligned} a_+(\theta,\eta) &= \frac{1}{\sqrt{2}}\Big(a_1(\theta,\eta) - ia_2(\theta,\eta)\Big) = Ub_+U^\dagger = \cosh\phi e^{i\psi}\ b_+ - \sinh\phi e^{-i\xi}\ b_-^\dagger \\ a_-(\theta,\eta) &= \frac{i}{\sqrt{2}}\Big(a_1(\theta,\eta) + ia_2(\theta,\eta)\Big) = iUb_-U^\dagger = i\Big[\cosh\phi e^{i\psi}\ b_- - \sinh\phi e^{-i\xi}\ b_+^\dagger\Big], \end{aligned} \tag{3.47}$$

where $b_+ = \frac{1}{\sqrt{2}}(b_1 - ib_2)$, $b_- = \frac{1}{\sqrt{2}}(b_1 + ib_2)$. These new Fock space operators also satisfy -

$$[a_\pm(\theta), a_\pm^\dagger(\theta)] = 1\ , [a_+(\theta), a_-(\theta)] = 0 = [a_+^\dagger(\theta), a_-^\dagger(\theta)] \tag{3.48}$$

Introduction of these operators finally renders the Hamiltonian diagonal -

$$H = \hbar\omega_+\left(a_+^\dagger(\theta,\eta)a_+(\theta,\eta) + \frac{1}{2}\right) + \hbar\omega_-\left(a_-^\dagger(\theta,\eta)a_-(\theta,\eta) + \frac{1}{2}\right); \omega_\pm = \omega\kappa' \mp \gamma \tag{3.49}$$

This is a system of decoupled oscillators with an underlying commutative phase-space where the characteristic frequencies are easily identified as $\omega_\pm$. Note that the spectrum of this system matches exactly with [84, 85]. The ground state is now defined as the tensor product of individual ground states as -

$$|0,0;\theta,\eta> \ = |0;\theta,\eta>_+ \otimes |0;\theta,\eta>_- \quad ; \ a_\pm(\theta,\eta)|0;\theta,\eta>_\pm = 0 \tag{3.50}$$

It then follows from above that the ground state also satisfies the following relation -

$$a_\pm(\theta,\eta)\,|0,0;\theta,\eta\rangle = 0 \implies b_\pm(U^\dagger\,|0,0;\theta,\eta\rangle) = 0. \tag{3.51}$$

And therefore we have,

$$|0,0;\theta,\eta\rangle = U\,|0,0;\theta=0,\eta=0\rangle \tag{3.52}$$

which relates the ground state of 2D harmonic oscillator in the NC phase space with that of 2D commutative harmonic oscillators. Also, the ground state density matrix of the phase space NC oscillator can be expressed as,

$$\rho(\theta,\eta) = |0,0;\theta,\eta\rangle\,\langle 0,0;\theta,\eta| = U\rho(\theta=0,\eta=0)U^{\dagger} \tag{3.53}$$

Moreover, the eigenbasis corresponding to the commutative number operator $(b_+^\dagger b_+ + b_-^\dagger b_-)$ can be defined as a tensor product decomposition into commutative subsystems, $\mathcal{H} = \mathcal{H}_+ \otimes \mathcal{H}_-$:

$$\mathcal{H} = \text{span}\{|n_+;\theta=0\rangle_+ \otimes |n_-;\theta=0\rangle_- = \frac{\left(b_+^\dagger\right)^{n_+}\left(b_-^\dagger\right)^{n_-}}{\sqrt{n_+!}\sqrt{n_-!}}\,|0,0;\theta=0\rangle\}_{n_+,n_-=0}^{\infty} \tag{3.54}$$

Using completeness relation of the state vectors $|n_+,n_-;\theta=0,\eta=0\rangle := |n_+;\theta=0,\eta=0\rangle_+ \otimes |n_-;\theta=0,\eta=0\rangle_- \in \mathcal{H}$ in (Eq. (3.52)), we have the ground state of the original Hamiltonian (Eq. (3.5)) as,

$$\begin{aligned}
|0,0;\theta,\eta\rangle &= \sum_{n_\pm=0}^{\infty} |n_+,n_-;\theta=0=\eta\rangle\,\langle n_+,n_-;\theta=0=\eta|\,U\,|0,0;\theta=0=\eta\rangle \\
&= \sum_{n_\pm=0}^{\infty} d_{n_+,n_-}(\theta,\eta)\;|n_+;\theta=0=\eta>_+ \otimes\, |n_-;\theta=0=\eta>_-
\end{aligned} \tag{3.55}$$

where the quantities $d_{n_+,n_-}(\theta,\eta)$ are called Wigner d- matrix elements,

$$d_{n_+,n_-}(\theta,\eta) = \langle n_+,n_-;\theta=0=\eta|\,U\,|0,0;\;\theta=0=\eta\rangle \tag{3.56}$$

These coefficients $d_{n_+,n_-}(\theta,\eta)$ can now be calculated easily for which it will be convenient to re-express U in terms of $b_\pm$ (for details see equation (Eq. (6.5)) in Appendix 1) to find:

$$d_{n_+,n_-}(\theta,\eta) = (-1)^{n_+}\mu^{n_+}\delta_{n_+,n_-} \tag{3.57}$$

Finally, we arrive at the desired form of the normalised ground state by replacing this $d_{n_+,n_-}(\theta,\eta)$

in (Eq. (3.55)) as -

$$|0,0;\theta,\eta>=\sqrt{1-|\mu|^2}\sum_{n=0}^{\infty}\ (-1)^n\mu^n|n;\theta=0,\eta=0>_+\otimes|n;\theta=0,\eta=0>_- \tag{3.58}$$

where μ is a complex number and is given by, $\mu=\tanh\phi\ e^{-i(\psi+\xi)}$ where ϕ,ψ and ξ have been previously defined in (Eq. (3.43)). The following is the corresponding ground state density matrix:

$$\begin{aligned}\rho &= |0,0;\theta,\eta><0,0;\theta,\eta| \\ &= (1-|\mu|^2)\sum_{n,m=0}^{\infty}(|\mu|)^{n+m}|n;\theta=0=\eta>_+<m;\theta=0=\eta|\otimes\ |n;\theta=0=\eta>_-<m;\theta=0=\eta|\end{aligned} \tag{3.59}$$

In terms of states of the commutative planar oscillator, these are the desired expressions of the ground state $|0,0;\theta,\eta>$ and the corresponding density matrix. Now, as the purity of a state is basis-independent, this is still a pure state, and the associated entropy vanishes. However, these two 1D harmonic oscillators are entangled, and the deformation terms in (Eq. (3.36)) produce the corresponding entanglement entropy. To show this, we will concentrate on sub-system '+' and perform partial tracing over sub-system '$-$' to produce the reduced density matrix for the subsystem '+' as:

$$\rho_+\equiv\rho_r=Tr_-\rho=(1-|\mu|^2)\sum_{n=0}^{\infty}\ (|\mu|^2)^n\ |n;\theta=0=\eta>_+<n;\theta=0=\eta| \tag{3.60}$$

which is clearly a mixed state density matrix as $\rho_r^2\neq\rho_r$ unlike the total density matrix ρ in (Eq. (3.59)). The eigenvalues p_n of this reduced density matrix ρ_r can be simply read-off in this manner as,

$$p_n=(1-|\mu|^2)|\mu|^{2n} \tag{3.61}$$

As a result, the von-Neumann entropy of this subsystem turns out to be -

$$\begin{aligned} S = & -Tr(\rho_r \log \rho_r) \\ = & -\sum_{n=0}^{\infty} p_n \ \log p_n \\ = & -\log\Big(1-|\mu|^2\Big) - \frac{|\mu|^2 \log |\mu|^2}{1-|\mu|^2} \end{aligned} \tag{3.62}$$

where,

$$|\mu| = \tanh\phi = \frac{1-\kappa'}{1+\kappa'} \tag{3.63}$$

3.2.1 Different limits of the entropy expression

Firstly, it is important to mention at this stage that all the equations derived in this section have a smooth ($\theta \to 0$ or $\eta \to 0$) limit enabling us to recover results of section 3.1 or the case of only momentum noncommutative space. For an example, on taking the momentum NC parameter η to be zero, the function μ (Eq. (3.63)) becomes real and equal to σ. So we recover (Eq. (3.31)) from the most general entropy expression (Eq. (3.62)). Moreover, in particular, for this limit of $\eta = 0$, we also have,

$$|0,0;\theta\rangle = U\,|0,0;\theta = 0\rangle\,, \tag{3.64}$$

which shows that the vacuum of the 2D noncommutative oscillators is nothing but a squeezed vacuum with respect to the commutative oscillator vacuum. This will be useful when we discuss the generalized Landau problem in the next section 3.3.

On the contrary if we allow the spatial noncommutativity parameter θ to be zero, then we end up in a similar expression as (Eq. (3.31)) for the entropy in such a case but where κ is now given by $\kappa = \sqrt{1+\frac{\eta^2}{4m^2\omega^2\hbar^2}}$. Moreover, we can observe that when the parameters θ and η both go to zero, the entropy vanishes. This is to be expected, as the system then becomes a two-dimensional set of independent oscillators with no interaction, resulting in zero entropy. Thus the expression Eq. (3.62) is general. Furthermore, the form of the entropy expression shows that the system can be ascribed an effective non-zero temperature. Following similarly as in the preceding section, the entropy can be thought of as the thermal entropy of an ensemble of identical 1-D oscillators with frequency ω kept in contact with a thermal bath at temperature $\frac{1}{T} = -\frac{2k_B}{\hbar\omega}\log|\mu|$. Noncommutativity might thus be understood as the cause of an emergent temperature in this sense.

3.2.2 A realization of non-linearity by noncommutativity: A qualitative discussion

Here, we discuss qualitatively the role of squeezing transformations and how the set of calculations presented earlier in this section may be relevant in the context of quantum optics. This is motivated from an intriguing mathematical similarity between these aspects in the two systems [1]. In particular, note that (Eq. (3.46)) is an U(2) transformation (i.e. a beam splitter like transformation) from the set of two input modes $a_1(\theta,\eta), a_2(\theta,\eta)$ to the set of normal modes $a_+(\theta,\eta), a_-(\theta,\eta)$ - both sets being considered in the background commutative plane satisfying standard Heisenberg algebra (Eq. (3.2)), as we have formulated in terms of an effective commutative theory. Now the non-classical nature of the input states results from (Eq. (3.44)) and hence as a result of the transformations in (Eq. (3.47)), the two-mode output state produced is nothing but the tensor product of two single-mode squeezed vacuum states, which will, in turn, become an entangled two-mode state. Thus we see that the transformation from the set of operators b_+, b_- to the set $a_+(\theta,\eta), a_-(\theta,\eta)$ via (Eq. (3.44)) maps the two-mode vacuum state $|0,0;\theta=0,\eta=0>$ to a two-mode entangled state (with any given value of θ and η) in the commutative plane. Note that the seed of the aforesaid entanglement lies in the existence of non-zero values of the non-commutativity parameters θ or η. A brief description of a process of relating the squeezing parameter (for single-and two-mode squeezing) with non-linear properties of crystals, used for generating squeezing will be done here. Details of computations relating to squeezing transformations have been discussed in Appendix 1.

[1] The scales of the noncommutative parameters θ and η should now be modified appropriately, and they should obviously be several orders of magnitudes larger than the one anticipated in quantum gravity scale.

Squeezing can be generated through various methods: using (i)parametric down conversion (three-wave or four-wave mixing), (ii)photon emissions by transitions between pairs of different energy levels of the atom, or (iii)optical fibres (three-wave mixing), etc. Non-linearity (i.e. non-linear dependence of polarization of light inside the crystal on the incoming electric field) is a crucial aspect and is at the heart of each of these methods [86].

In the interaction picture, the Hamiltonian for non-degenerate parametric amplification in a fully quantum-mechanical description is given by [86]-

$$H = \hbar g(a_s^\dagger a_i^\dagger b_p + a_s a_i b_p^\dagger) \tag{3.65}$$

where g is the coupling constant, which is a function of second order non-linearity parameter $\chi^{(2)}$.

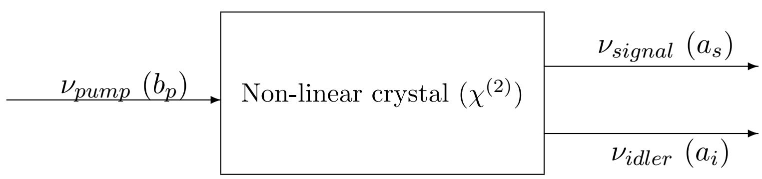

Figure 3.1: A schematic diagram of three-wave mixing in parametric amplification

One generally considers the pump field to be classical in parametric amplification so that we can replace b_p by $\beta_p e^{i\phi_p}$ where β_p is the amplitude and ϕ_p is the phase of coherent pump field. Under this situation, we have

$$H = \hbar g\beta_p(a_s^\dagger a_i^\dagger e^{-i\phi_p} + a_s a_i e^{i\phi_p}) \tag{3.66}$$

This condition is valid when: $gt \longrightarrow 0, \beta_p \longrightarrow \infty$ such that $g\beta_p t =$ constant, where t is the time in the travelling wave reference frame.

As a matter of fact, in the case of squeezing (single as well as two-mode), the non-linearity parameter $\chi^{(2)}$ of the crystal is actually a function of the coupling parameter g. An explicit form of g, in the case of two-mode squeezing via non-degenerate parametric amplification in the context of three wave mixing (as depicted in Figure 1) is given by [86–88] -

$$g\beta_p t = \chi^{(2)}\beta_p t = \chi^{(2)}|\Omega|\frac{nl}{c} \tag{3.67}$$

where $|\Omega| \equiv \beta_p$ =amplitude of input pump, n =refractive index of the crystal, l =length of the crystal, c =speed of light in vacuum. Thus, in this case of three-wave mixing, the associated squeezing parameter for two-mode squeezing, is given by:

$$|s| = \chi^{(2)}|\Omega|t = \chi^{(2)}|\Omega|\frac{nl}{c} \tag{3.68}$$

Hence the amount of squeezing s depends linearly on the crystal non-linearity parameter $\chi^{(2)}$. As here $\beta_p \neq 0$ and $l \neq 0$, therefore, $s = 0$ iff $\chi^{(2)} = 0$. It should be noted that the parametric amplification mechanism shown in Figure (1) is probabilistic (with low success probability). As a result, the 'effective' Hamiltonian in Eq. (3.65) is only relevant when the parametric amplification process is successful, which occurs when certain conservation laws (energy conservation, momentum conservation, etc.) remain true [86–88]. The Bogolyubov transformation (Eq. (3.42)) can be thought of as a squeezing transformation (Eq. (6.3)) [see Appendix 1] with a squeezing parameter s that depends on the noncommutative parameters θ and η, and as a result, the non-linear parameter $\chi^{(2)}$ gets naturally related to the parameters θ and η (this, as discussed above, holds as the two-mode squeezing happens due to the presence of the parameter $\chi^{(2)}$). When the parametric amplification process in Figure 1 succeeds, the structure of phase-space inside a non-linear crystal, perhaps, may be modelled as non-commutative in nature. We also want to point out that the foregoing generation of squeezing was only considered via parametric amplification. As previously stated, non-linearity is required for all other squeezing resources.

It is worthwhile to mention here that if $\eta = 0$ and $\theta \neq 0$ as in previous section 3.1, we have a two-mode squeezing transformation with a real squeezing parameter $s = \frac{1}{4}\log\left(1 + \frac{m^2\omega^2\theta^2}{4\hbar^2}\right)$ in contrast to the complex squeezing parameter, $s = \phi e^{i(\psi+\xi)}$ when both types of noncommutativity exist simultaneously. Furthermore, when both $\theta \neq 0$ and $\eta \neq 0$, the squeezing amount increases as compared to that in the presence of a single kind of noncommutativity, since the extra contributions are positive-definite (see Appendix 1, Eq. (6.2)). As a result, noncommutativity serves as a resource for squeezing.

3.3 Landau problem and analogue Unruh effect

It is well-known that the mechanical momentum components do not commute, in contrast to the components of canonically conjugate momentum, in the Landau problem. However in this section, we are going to first establish a correspondence between the Landau problem in the presence of harmonic potential and the two-dimensional NC oscillator system studied in section 3.1. This will allow us to realize the entropic dynamics without considering spatial noncommutativity at the first place, as the commutative coordinates may just be theoretical constructs in an ambient NC space and can be difficult to realize in the experimental context (laboratory). To that end, we recall the Hamiltonian of a charged particle moving in a plane with a normal external magnetic field being applied and in presence of an additional 2D harmonic potential (with frequency ω) takes the following form in the symmetric gauge ($A_i = -\frac{B}{2}\epsilon_{ij}q_j$) :

$$\begin{aligned} H &= \frac{(\pi_i - eA_i)^2}{2m} \\ &= \frac{1}{2m}(\pi_1^2 + \pi_2^2) + \frac{1}{2}m\Omega^2(q_1^2 + q_2^2) - \frac{\omega_c}{2}(q_1\pi_2 - q_2\pi_1) \end{aligned} \tag{3.69}$$

where $\omega_c = \frac{eB}{m}$ is the cyclotron frequency, $\Omega = \omega\lambda$, $\lambda = \sqrt{1 + \frac{\omega_c^2}{4\omega^2}}$. Note here π_i is the canonical momentum satisfying the commutative Heisenberg algebra (Eq. (4.4)) and e is the electric charge.

Now notice that the Hamiltonian (Eq. (3.5)) of the latter system can be unitarily transformed into the following form :

$$H \to H' = \tilde{U} H \tilde{U}^\dagger = \frac{1}{2m}(\pi_1^2 + \pi_2^2) + \frac{1}{2} m\omega^2 \kappa^2 (q_1^2 + q_2^2) - \frac{m\omega^2 \theta}{2\hbar}(q_1 \pi_2 - q_2 \pi_1) \tag{3.70}$$

where the unitary operator $\tilde{U}$ is the exponential of one of the $SU(1,1)$ generators and is given by $\tilde{U} = e^{-\frac{i \log(\kappa)}{2\hbar}(q_i \pi_i + \pi_i q_i)}$.

The Hamiltonians (Eq. (3.70)) and (Eq. (3.69)) are of similar form and become equivalent upon identifying:

$$\theta = \frac{eB\hbar}{m^2 \omega^2} \tag{3.71}$$

As a result, the parameter λ is associated with κ (Eq. (3.7)). In this scenario, the chargeless ($e = 0$) limit of the Landau problem, for a given constant value of magnetic field in presence of oscillatory interaction, yields the commutative limit of the Moyal plane. This establishes the correspondence between the Landau problem with harmonic potential and the two-dimensional NC oscillator system covered in section 3.1 and the associated chargeless/commutative limits. Consider a neutral particle ($e = 0$) that would not have been aware of the magnetic field's presence. This corresponds to $\theta = 0$, the commutative limit in the previous subsection. This neutral particle's Hamiltonian will then be:

$$H = \frac{1}{2m}\vec{\pi}^2 + \frac{1}{2} m\omega^2 \vec{q}^2 \tag{3.72}$$

The proper Fock space operators for this isotropic harmonic oscillator system in the plane are the conventional 2D harmonic oscillator ladder operators (b_i) presented in (Eq. (3.8)). Given that the mapping between the two physical descriptions has been established, we will now work with the Hamiltonian (Eq. (3.69)), which may be diagonalized using the canonical Bogolyubov transformation, as we did in the previous subsection -

$$a_i(B) = U b_i U^\dagger = \cosh\phi \; b_i - \sinh\phi \; b_i^\dagger, \; with \; \phi = \log\left(\sqrt{\lambda}\right) \tag{3.73}$$

where $U = e^{-\frac{\phi}{2}(b_j^2 - b_j^{\dagger 2})}$ is the squeezing operator. Note here that the squeezing parameter ϕ is real here unlike in the case of phase-space noncommutativity (Eq. (3.44)). We have obviously, $[a_i(B), a_j^\dagger(B)] = \delta_{ij}$. The diagonalized form of the Hamiltonian (Eq. (3.69)) finally turns out to be:

$$H = \hbar\Omega_+ \left(a_+^\dagger(B)a_+(B) + \frac{1}{2}\right) + \hbar\Omega_- \left(a_-^\dagger(B)a_-(B) + \frac{1}{2}\right) \tag{3.74}$$

where the characteristic frequencies $\Omega_\pm$ are given by, $\Omega_\pm = \Omega \pm \frac{1}{2}\omega_c$. The normal mode operators $a_\pm(B)$ were obtained by applying a $U(2)$ transformation to $a_i(B)$, much as in (Eq. (3.46)) to get -

$$\begin{pmatrix} a_1(B) \\ a_2(B) \end{pmatrix} \longrightarrow \begin{pmatrix} a_+(B) \\ a_-(B) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} a_1(B) \\ a_2(B) \end{pmatrix} \tag{3.75}$$

while using (Eq. (3.73)), $a_\pm(B)$ and $b_i's$ are related as

$$a_+(B) = U b_+ U^\dagger \; ; \; a_-(B) = i U b_- U^\dagger \tag{3.76}$$

Inverting the above relations, we get-

$$b_+ = U^\dagger a_+(B) U = \cosh\phi \; a_+(B) + i\sinh\phi \; a_-^\dagger(B) \tag{3.77}$$

$$b_- = -iU^\dagger a_-(B) U = \sinh\phi \; a_+^\dagger(B) - i\cosh\phi \; a_-(B) \tag{3.78}$$

The neutral particle's ground state is given by: $b_\pm|0,0>_o = 0$, from which it follows that-

$$a_\pm(B)(U|0,0>_o) = 0 \implies |0,0;B> = U|0,0>_o \tag{3.79}$$

And therefore we have,

$$|0,0>_o = U^{\dagger}|0,0;B> \tag{3.80}$$

This allows one to write down the neutral particle's ground state $|0,0>_o$ as a linear combination of tensor products of the charged particle's modes '±' as :-

$$|0,0>_o = \sqrt{1-\zeta^2}\sum_{n=0}^{\infty}\ (-1)^n\zeta^n|n;B>_+ \otimes |n;B>_- \tag{3.81}$$

where, $\zeta = \frac{\lambda-1}{\lambda+1}$. The subscript 'o' designates the neutral particle's ground state. In comparison to (Eq. (3.58)), this expansion (Eq. (3.81)) is in the opposite direction in that the commutative (neutral) vacuum has been expanded in terms of states of the noncommutative (charged) oscillator, whereas in (Eq. (3.58)), the noncommutative vacuum has been expanded in terms of states of the commutative oscillator

Now, let the applied magnetic field be very strong as compared to the harmonic interaction i.e. $\omega_c >> \omega$. In this situation, we have:

$$\Omega_+ \simeq \omega_c + \frac{\omega^2}{\omega_c}\ ;\ \ \Omega_- \simeq \frac{\omega^2}{\omega_c} \tag{3.82}$$

obtained after keeping terms only upto first order in $\frac{\omega^2}{\omega_c^2}$. Because there is a large energy gap in the system, the higher states will be effectively inaccessible for a low energy particle. As a result, we form the reduced density matrix by tracing out the higher energy modes $|n;B>+$ from the total density matrix as follows:

$$\rho_r = Tr_+\rho = (1-\zeta^2)\sum_{n=0}^{\infty}\ (\zeta^2)^n\ |n;B>_-<n;B| \tag{3.83}$$

which has the same form as (Eq. (3.60)) with $|\mu|$ replaced by ζ ($|\mu| \to \zeta$). Therefore, if one continues the computations exactly as described in the previous section, one can assign an entropy expression

:

$$S = -\log\left(1-\zeta^2\right) - \frac{\zeta^2 \log \zeta^2}{1-\zeta^2} \tag{3.84}$$

and an effective temperature,

$$\frac{1}{T} = -\frac{2k_B}{\hbar\omega}\log\zeta \tag{3.85}$$

where we simply replaced $|\mu|$ with ζ in (Eq. (3.62)) to achieve these formulas, the neutral particle's Fock vacuum $|0>_o$ now appears as an entangled state when represented as a sum of tensor products of multiple Fock states of the charged particle.

At this point, we notice a strong connection between the Unruh effect in 1+1 D quantum field theory (see [89] for a review) and the physical essence of the results deduced for the "generalised Landau problem". The analogy is drawn from both physical and mathematical considerations, and to elucidate it, we must first recall that the discretized version of the vacuum Minkowski state $|0>_M$ looks like this:

$$|0>_M = \prod_{i=1}^{\infty}\sqrt{1-e^{\frac{-2\pi\omega_i}{a}}}\sum_{n_i=0}^{\infty} e^{\frac{-n_i\pi\omega_i}{a}} \; |n_i; R> \otimes |n_i; L> \tag{3.86}$$

where a is the acceleration of the Rindler observer and ω_i being the set of discrete frequencies, when expanded in terms of the tensor product of Fock spaces, associated with the Rindler observers in the left and right wedges.

The two modes $\Omega_{\pm}$ are now energetically "decoupled" (Eq. (3.82)) in the large magnetic field limit due to the appearance of a high energy gap in the system. This energy gap represents two causally detached worlds and parallels the role of null horizons in detaching the left and right Rindler wedges. Despite the fact that these wedge states are discontinuous, they produce an entanglement. On tracing out the unrelated (disjointed) left wedge, an observer in the right Rindler wedge discovers the Minkowski vacuum as a thermal bath of particles at the required temperature, $T = \frac{a}{2\pi k_B}$ where the reduced density matrix is provided by:

$$\rho_R = \prod_{i=1}^{\infty}\left((1-e^{\frac{-2\pi\omega_i}{a}})\sum_{n_i=0}^{\infty} e^{\frac{-2\pi n_i\omega_i}{a}} |n_i;R><n_i;R|\right) \tag{3.87}$$

Similarly we can integrate out the inaccessible higher energy modes Ω_+ to find that the neutral vacuum is observed as a thermal mixed state by the charged observer. This observation can be further illuminated by noting the fact that the mean occupation number $a_-^\dagger(B)a_-(B)$ in the state ρ_r (Eq. (3.83)) of the neutral particle takes the form of Bose-Einstein distribution:

$$< a_-^\dagger(B)a_-(B) >_{\rho_r} = \frac{1}{e^{\frac{\hbar\omega}{k_B T}}-1} \tag{3.88}$$

As a result, a simple examination of the structures of equations (Eq. (3.81)) and (Eq. (3.86)) shows that these two systems have a natural one-to-one correspondence: the three states $|0,0>_o$, $|n;B>_+$ and $|n;B>_-$ in (Eq. (3.81)) mimics the states $|0>_M$, $|n_i;R>$ and $|n_i;L>$ in (Eq. (3.86)) , and the parameter ζ in (Eq. (3.81)) corresponds to the acceleration a of the Rindler observer. Therefore, the neutral particle vacuum (counterpart of the vacuum in inertial frame $|0>_M$) is perceived to be a thermal state from the perspective of the charged observer (accelerating observer). Thus it defines the analogue Unruh effect in "generalized" Landau systems. Note that the expression of entropy (Eq. (3.84)) does not depend upon which mode one traces out. It becomes imperative to point out that our current demonstration, which makes use of the Bogolyubov transformation (Eq. (3.73)) as well as (Eq. (3.80)) involving the SU(1,1) element U, provides the necessary fingerprint to understand the concept of thermalization and associated entanglement as in the Unruh problem, as first explicitly shown in [90], see also [91].

We also have :

$$|0,0;\theta=0>\equiv|0,0>_o=\sqrt{1-\sigma^2}\sum_{n=0}^{\infty} \sigma^n|n;\theta>_+\otimes|n;\theta>_- \tag{3.89}$$

where the R.H.S. represents the states of an observer in the effective noncommutative world, as following from (Eq. (3.81)) and the relationship we mentioned earlier (Eq. (3.71)).

3.4 Remarks

In this chapter, we saw that when a 2D oscillatory system is put in deformed spaces, such as the Moyal plane or noncommutative phase spaces, a non-zero entropy emerges in the ground state. For a class of Hamiltonian systems proposed by t' Hooft [10], our approach may provide an approximation of the loss of information due to quantization. Then, we exploited the equivalence of the Landau system with oscillatory interactions and the 2D noncommutative oscillator system to compute the entropy of neutral oscillators as observed by a charged harmonic oscillator in the presence of a magnetic field in section 3.3. To put it another way, the charged particle perceives the neutral particle's vacuum as a thermal state. We had also shown how, in the case of the "generalised Landau system", our result has an intriguing resemblance to the Unruh effect. Thus, for the "generalised Landau problem", an Unruh-like physics has been discovered. Such effects in cold Rydberg atom systems where position NC arises [31] will be fascinating to test. Furthermore, our findings imply that an anyonic charged particle perceives the spinless chargeless particle's vacuum as a mixed state. Finally, we provide a rudimentary discussion of the appearance of two-mode squeezing in a system of two oscillators as a result of phase-space NC in a non-linear medium, which causes the squeezing effect. Probing the impacts of deformation of quantum systems has been a difficult task and this present study has opened a new avenue to investigate these effects where some fascinating studies have been taken up very recently, for example in [92] and references therein.

Chapter 4

Noncommutativity from curvature

In the last chapter, we introduced phase space noncommutativity where the momenta do not commute in adition to the position coordinates of a particle. Here, in this chapter, we take up the case, where solely the momentum space noncommutativity is present and discuss its effects on the concerned systems. It will be shown that momentum space noncommutativity does lead to a deformation in the energy spectrum even for a free particle (nonrelativistic/relativistic), unlike for coordinate noncommutativity where external two-dimensional potentials are indispensable in order to have deformation in spectrum. Now, it has been well-known since the advent of Einstein's geometrical formulation of gravity [93] that curvatures in space give rise to a noncommutation among the translation generators of the space [94, 95]. In this sense, therefore momentum noncommutativity has a simple interpretation as an effect of curvature in position space of a manifold. Here, we will discuss the effect of momentum-momentum noncommutativity on the quantum mechanical statistical properties of an electron gas. Then we will discuss a plausible implication of these altered statistical quantities on white dwarf stars. Remarkably so, the mass-limit of white dwarfs formulated by S. Chandrasekhar [96] has been one of the most celebrated discoveries in twentieth century astrophysics. This limiting mass is used in order to regard the type Ia supernovae (SNeIa) as a standard candle; a white dwarf on approaching its limiting mass of $1.4M_{\odot}$, called Chandrasekhar mass-limit, SNeIa is likely to be triggered [96]. Now, some peculiar, over-luminous SNeIa such as

SN 2003fg, SN 2006gz, SN 2009dc [44,45] have been observed with unusually high luminosity and low ejecta velocity, suggesting the progenitor mass to be higher of around $2.8M_{\odot}$. Some examples of such peculiar over-luminous SNeIa are SN 2003fg, SN 2006gz, SN 2007if, SN 2009dc. Certain proposals have been made involving highly magnetized white dwarfs [97,98] to explain this violation, these models however suffer from severe stability problems [99,100].

As a matter of fact, we now know that curvature in position space leads to a noncommutativity in the momentum variables. In an analogous manner this implies that curved momentum spaces can lead to noncommutativity among the position variables by invoking "principle of reciprocity". Indeed in [15], S. Majid has used this principle to show that quantum phase space should contain quantum spacetime and there should be noncommutativity both in the configuration space variables as well as in momentum variables in order to describe quantum gravity. At this juncture, here we consider existence of noncommutativity among momentum components by first taking it as a postulate and investigate the emergence of new physics in the quantum mechanical statistical properties of a gas living in such a space. It will then be argued that our consideration of noncommutativity is not adhoc but comes from a natural argument relating curvature with momentum noncommutativity. The idea is to consider noncommutativity at the high density regime of non-magnetized white dwarfs. The matter inside white dwarf is essentially a degenerate electron gas and it is this electron degeneracy pressure that balances the star by opposing gravity inside the star throughout. We will be exploring the effect of momentum noncommutativity in the equation of state (EoS) and finally its ramifications on the mass limit of white dwarfs.

Being a compact object, the typical densities of white dwarfs are in the range of $10^6 - 10^{10} g/cc$, however, these associated length scales are still very very far from that of the quantum gravity scale. Thus one can ignore spatial noncommutativities in the present formulation and one is motivated to figure out the consequences of momentum-momentum noncommutativity only, arising from curved position space in the matter of white dwarf stars, even if not being in the quantum gravity regime.

For simplicity of our present treatment, we shall also neglect the inter-particle Coulomb interaction in a white dwarf star in the light of the arguments in [101].

4.1 Formalism: Relativistic electron gas

Considering relativistic electrons of mass m_e moving in a three-dimensional space where the three momentum coordinates $\hat{\pi}_1$, $\hat{\pi}_2$ and $\hat{\pi}_3$ do not commute but the position coordinates $\hat{q}_i$ commute :

$$[\hat{\pi}_i, \hat{\pi}_j] = i\eta_{ij} \ , \ \ [\hat{q}_i, \hat{q}_j] = 0 \ , \ \ [\hat{q}_i, \hat{\pi}_j] = i\hbar\delta_{ij} \ ; \ i, j = 1, 2, 3.$$

where η_{ij} is a 3×3 anti-symmetric matrix. Now any odd antisymmetric matrix can be brought to a block diagonal form wherein the bottom right block becomes a null matrix. This makes the third component of momentum to commute with the other two [102, 103]. Therefore now we have the NC Heisenberg algebra (NCHA) to be

$$\begin{aligned} &[\hat{x}_j, \hat{x}_k] = 0, \ \ [\hat{x}_j, \hat{p}_k] = i\hbar\delta_{jk} \\ &[\hat{p}_a, \hat{p}_b] = i\eta\epsilon_{ab}, \quad [\hat{p}_a, \hat{p}_z] = 0, \text{for } a, b = 1, 2. \end{aligned} \tag{4.1}$$

where subscripts 1 and 2 respectively imply x and $y-$components of respective variables. Here η is the momentum NC parameter. As it will become transparent subsequently that the spectrum remains unaffected by the aforesaid transformation of the operators, as spectrum will be shown to depend on the eigenvalues of the η matrix and such a kind of rotation does not change the eigenvalues of the η matrix.

4.1.1 Energy spectrum

We now move on to study the motion of an ultra-relativistic electron in such a background where only two of the momentum components are being noncommutative in nature. To that end, we first write down the Dirac equation of these relativistic electrons satisfying (Eq. (4.1)),

$$\hat{H}\psi = i\hbar\frac{\partial\psi}{\partial t} = E\psi \quad ; \quad \psi = \begin{pmatrix} \phi \\ \\ \chi \end{pmatrix} \tag{4.2}$$

where ϕ and χ both being 2-component spinors and the Dirac Hamiltonian is given by

$$\hat{H} = \vec{\alpha}.\vec{\hat{p}}c + \beta m_e c^2, \tag{4.3}$$

where c is the speed of light.

As the Hamiltonian involves non-commuting momenta, the above equation does not yield the usual relativistic dispersion ($E = \sqrt{p^2c^2 + m_e^2c^4}$) for the free electrons. To solve the spectrum of these ultra-relativistic electrons satisfying the algebra (Eq. (4.1)), instead we form an equivalent commutative description of the noncommutative theory by employing the previously discussed generalised Bopp shift (Eq. (3.35)) after setting $\theta = 0$, which relates the NC operators $\hat{x}_j$, $\hat{p}_j$ following equation (Eq. (4.1)) to ordinary commutative operators x_j, p_j, satisfying the usual Heisenberg algebra:

$$[x_j, p_k] = i\hbar\delta_{jk} \ , \quad [p_1, p_2] = 0. \tag{4.4}$$

As always, we are denoting NC operators with the hat notation and commutative operators without hat. The two sets of operators are related as,

$$\hat{p}_j = p_j + \frac{\eta}{2\hbar}\epsilon_{jk}x_k \tag{4.5}$$

From Eq. (4.2) and Eq. (4.3), we get the following pair of equations:

$$
\begin{aligned}
(E - m_e c^2)\phi &= \vec{\sigma}.\vec{\hat{p}}c\ \chi \\
\text{and,}\quad (E + m_e c^2)\chi &= \vec{\sigma}.\vec{\hat{p}}c\ \phi
\end{aligned} \tag{4.6}
$$

Now on combining the above equations, we obtain the following -

$$
\begin{aligned}
(E^2 - m_e^2 c^4) &= (\vec{\sigma}.\vec{\hat{p}})^2 c^2 \\
&= \hat{p}^2 c^2 + i\vec{\sigma}.(\vec{\hat{p}} \times \vec{\hat{p}})c^2 \\
&= \left[(p_x^2 + p_y^2) + \frac{\eta^2}{4\hbar^2}(x^2 + y^2) + \frac{\eta}{\hbar}(yp_x - xp_y) + p_z^2 - \sigma_z \eta\right] c^2
\end{aligned} \tag{4.7}
$$

Therefore, we have obtained an equivalent commutative Hamiltonian in terms of the commutative variables (quantum mechanical operators) which describes the original system defined over the NC momentum space.

In order to compute the spectrum of the electrons in such a NC spacetime, first of all we need to construct the ladder operators that will diagonalize the following part of right hand side of equation (Eq. (4.7)), which we denote as $\hat{H}'$ below,

$$
\hat{H}' = \left[(p_x^2 + p_y^2) + \frac{\eta^2}{4\hbar^2}(x^2 + y^2) + \frac{\eta}{\hbar}(yp_x - xp_y)\right] c^2 \tag{4.8}
$$

The ladder operators involving the commutative phase-space operators x, y, p_x, p_y are constructed as,

$$
\begin{aligned}
a_j &= \frac{1}{\sqrt{\eta}}\left(p_j - i\frac{\eta}{2\hbar}x_j\right), \\
a_j^\dagger &= \frac{1}{\sqrt{\eta}}\left(p_j + i\frac{\eta}{2\hbar}x_j\right)
\end{aligned} \tag{4.9}
$$

satisfying the commutation relations

$$[a_x, a_x^\dagger] = 1 = [a_y, a_y^\dagger]. \tag{4.10}$$

Furthermore, by using these operators to define the following pair of operators

$$\hat{a}_1 = \frac{a_x + i a_y}{\sqrt{2}}, \quad \hat{a}_2 = \frac{a_x - i a_y}{\sqrt{2}}, \tag{4.11}$$

which satisfy the commutation relations

$$[\hat{a}_1, \hat{a}_1^\dagger] = 1 = [\hat{a}_2, \hat{a}_2^\dagger], \tag{4.12}$$

the part $\hat{H}'$ given by equation (Eq. (4.8)) can be recast in the diagonal form as,

$$\hat{H}' = \eta(2\hat{a}_1^\dagger \hat{a}_1 + 1)c^2 \tag{4.13}$$

Therefore, on using equations (Eq. (4.7)) and (Eq. (4.13)), the total energy of the system is finally computed to be -

$$E^2(m) = p_z^2(m)c^2 + m_e^2 c^4 + 2m\eta c^2, \tag{4.14}$$

where m is a quantum number, and is given by $m = n_1$ for spin-up ($s = \frac{1}{2}$) when n_1 is the eigenvalue of the number operator $\hat{a}_1^\dagger \hat{a}_1$, and $m = n_1 + 1$ for spin-down ($s = -\frac{1}{2}$). Notice that the last term contributing to the energy in the expression above signifies the result of quantization of the levels and the energy contribution from the $x - y$ momenta are no more continuous as opposed to the case of free electrons moving in a three-dimensional commutative space. This occurs solely due to the noncommutative nature of the momenta components $\hat{p}_x, \hat{p}_y$. Having done with the spectrum, we move ahead to study the quantum mechanical statistical properties of these electrons in the next section.

4.2 Statistical properties

The deformation of the energy spectrum will affect the statistical properties of the electrons; the available density of states is going to be modified for the electrons in presence of this momentum space noncommutativity. To that end, let us consider the Fermi energy E_F of electrons for the mth level to be given as,

$$E_F^2(m) = p_F^2(m)c^2 + m_e^2c^4 + 2m\eta c^2 \tag{4.15}$$

For simplicity, we re-write the above in dimensionless form as,

$$\epsilon_F^2 = x_F^2(m) + 1 + 2m\eta_D \quad ; \quad \eta_D = \frac{\eta}{m_e^2c^2} \tag{4.16}$$

where $\epsilon_F = \frac{E_F}{m_e c^2}$ is the dimensionless Fermi energy, the dimensionless Fermi momentum $x_F(m) = \frac{p_F(m)}{m_e c}$ and, η_D is the dimensionless momentum noncommutative parameter which will be an important factor in our subsequent analysis.

As there is a quantization in the energy levels in the $x - y$ sector, the allowed range of the 3-momenta $\vec{p}$ now lies within a cylinder of radius $\sqrt{2m\eta}$, whose axis is along p_z in momentum space as can be easily inferred from Eq. (4.14). Thus, the number of electronic states available in the interval dp_z for a particular arbitrary value of m i.e. the modified density of state turns out to be:

$$D(p_z)dp_z = \frac{4\pi\eta}{h^3}dp_z \tag{4.17}$$

Hence, the electron number density and electron energy density at zero temperature are respectively given by :

$$n_e = \sum_{m=0}^{m_{max}} g_m D(p_z) p_F(m) = \sum_{m=0}^{m_{max}} \frac{4\pi m_e^3 c^3 \eta_D}{h^3} g_m x_F(m), \tag{4.18}$$

$$u = \frac{4\pi m_e^3 c^3 \eta_D}{h^3} \sum_{m=0}^{m_{max}} g_m \int_0^{x_F} E(m) dx(m), \tag{4.19}$$

where g_m is the degeneracy such that $g_m = 1$ for $m = 0$ and $g_m = 2$ for $m \geq 1$, where m in Eq. (4.18) and Eq. (4.19) is taken to be the largest integer $m < m_1 = (\epsilon_F^2 - 1)/2\eta_D$ for every ϵ_F and η_D. Therefore the pressure of the Fermi gas [104, 105] is calculated to be:

$$\begin{aligned} P &= n_e E_F - u \\ &= \sum_{m=0}^{m_{max}} \frac{2\pi m_e^4 c^5 \eta_D}{h^3} g_m \left[\epsilon_F x_F(m) - (1 + 2m\eta_D) \log \left(\frac{\epsilon_F + x_F(m)}{\sqrt{1 + 2m\eta_D}} \right) \right]. \end{aligned}$$

Finally, the mass density is given by -

$$\rho = \mu_e m_n n_e, \tag{4.20}$$

where μ_e is the mean molecular weight per electron and m_n is the mass of a neutron.

Having derived the energy spectrum and the modified quantum mechanical statistical properties for an electron gas exhibiting momentum noncommutativity, we now proceed ahead to find an application of the ideas developed in this chapter so far, in the context of matter inside white dwarfs stars where electron degeneracy pressure is known to play a very crucial role - for the existence of limiting mass of white dwarfs and the following consideration has been strongly motivated along the lines of [77].

4.3 As a curvature effect: Implications in white dwarfs

The noncommutative algebra (Eq. (4.1)) we are dealing here may rather be based on an intuitive physical picture and the origin of noncommutativity in momentum space can be argued heuristically in the following way. Primarily, the idea here is to consider the motion of a typical ultra-relativistic electron in the background of the gravitational field produced by the stellar material inside a white dwarf star, unlike that in Chandrasekhar's original analysis where the electrons were essentially treated to be free [96]. However like his analysis, we have assumed the potential to be infinitely large outside the star, thus trapping the electrons inside the surface. By considering the gravitational interaction on an electron due to the surrounding stellar matter, which is responsible for the effect of creating a curvature in a small region of the ambient space, we now model a test electron propagating under this gravitational background, i.e. propagating on the associated curved space [77,106]. At this stage, it will be worthwhile to mention that our main interest lies in the dynamics of ultra-relativistic electrons, simply because these electrons correspond to the existence of limiting mass of white dwarfs in Chandrasekhar's standard analysis, and for such ultra-relativistic electrons gravity effects are encoded in spatial curvature [107], [108]. Generically, this situation can be thought of as the stable circular motion of a test electron under the influence of gravity of the surrounding matter and the existence of bound orbits is governed by Laplace equation, both in Newtonian gravity and general relativity [109]. The corresponding motion can always be mapped to the motion of a free electron on the sphere as geodesic flows on the sphere is equivalent to the Kepler problem [110]. This is precisely the case we deal here. The aforesaid consideration closely conforms to the well-known observations made in the seminal work [77] in which it has been argued that there must be strong curvature at short length scales, which, averages to zero at large scales. Thus we are motivated to consider the effective motion of electrons on the surface of a sphere resulting in appearance of noncommutativity of momentum components in the two directions and a commutative momentum for the third direction. Having said this, we start our analysis explicitly.

The commutator of the covariant derivatives acting on a spinor (electrons) when written in the local orthonormal basis is given as [111,112] -

$$[\nabla_a, \nabla_b]\,\psi = \frac{1}{4} R_{abcd}\gamma^c\gamma^d\ \psi,$$

where ψ is the fermion wave function representing the electron, $\gamma^{c,d}$ are Dirac γ-matrices and R_{abcd} are the Riemann curvature tensor components in the local coordinates. Now for a space with constant scalar curvature R, which is $\frac{2}{r^2}$ for a sphere S^2, R_{abcd} are actually constants, being only proportional to the scalar curvature R and γ's are also constant matrices. If we now define

$p_a = -i\hbar\nabla_a$, we obtain

$$[p_a, p_b]\,\psi = i\frac{\hbar^2}{4r^2}S_{ab}\;\psi \tag{4.21}$$

where $S_{ab} = i[\gamma_a, \gamma_b]$. Therefore, the above commutation relation is of canonical type. Similar kind of spin-dependent noncommutative structures have also appeared previously in the literature [113–115]. On comparing equations (Eq. (4.1)) and (Eq. (4.21)), we see that the quantity $\frac{\hbar^2}{4r^2}$ plays the role of momentum noncommutativity parameter η.

We now assume that all the electrons are filled in the lowest Landau level, hence $m = 0$. The validity of this assumption will be discussed at the end of this section. Now for $m = 0$, on using (Eq. (4.18)) and (Eq. (4.20)), we can write the mass density as:

$$\rho = Qx_F(0), \tag{4.22}$$

$$\text{where,}\quad Q = \frac{4\pi\mu_e m_n m_e^3 c^3}{h^3}\eta_D. \tag{4.23}$$

and EoS reduces to

$$P = \frac{h^3}{8\pi\mu_e^2 m_n^2 m_e^2 c\eta_D}\left[\rho\sqrt{\rho^2+Q^2} - Q^2\log\frac{\rho+\sqrt{\rho^2+Q^2}}{Q}\right],$$

In the case of $x_F(0) >> 1$, which corresponds to $\rho^2 >> Q^2$, EoS further reduces to a simpler polytropic form as

$$P = \frac{h^3}{8\pi\mu_e^2 m_n^2 m_e^2 c\eta_D}\rho^2 \tag{4.24}$$

where the polytropic index is $n = 1$.

However, for the present case, also $x_F^2 = \epsilon_F^2 - 1 > 2m\eta_D$ which implies $\epsilon_F^2 = 2m_1\eta_D + 1$ where $0 \lesssim m_1 < 1$, particularly at the center and for ground level (analog of the lowest Landau level for the magnetic case) $\sqrt{\epsilon_F^2(0)-1} = x_F(0) = \sqrt{2m_1\eta_D}$. Therefore, at center

$$\rho = \rho_c = \frac{4\pi\mu_e m_n m_e^3 c^3}{h^3}\eta_D^{3/2}\sqrt{2m_1}, \tag{4.25}$$

when m_1 can have any value below unity for all electrons to be in the ground level. On eliminating η_D from equations (Eq. (4.24)) and (Eq. (4.25)), one obtains

$$P = K_{nc}\rho^{4/3} \quad , \qquad \text{where} \tag{4.26}$$

$$\begin{aligned} K_{nc} &= \frac{hc}{2}\left(\frac{m_1}{2\pi\mu_e^4 m_n^4}\right)^{1/3} \\ &= 1.1\times 10^{15} m_1^{1/3} \end{aligned} \tag{4.27}$$

(Eq. (4.26)) looks very similar to the EoS as is obtained from Chandrasekhar's analysis except for the fact that the constant of proportionality K_{nc} is different. In the considered scenario, this important constant has been augmented as compared to its value in the standard case. This is one of the most important results of the present chapter and will have a key bearing on our subsequent results.

Fixing noncommutativity parameter

From equation (Eq. (4.25)), let us consider the central density of the white dwarf to be given by,

$$\rho_c = \frac{2\times 10^{10}}{V} \quad \text{gm/cc}, \tag{4.28}$$

where V is a parameter allowing to change the central density, we can fix η_D at the center of the star as,

$$\eta_D = \left(\frac{2\times 10^{10} h^3}{4\pi\mu_e m_n m_e^3 c^3\sqrt{2m_1}V}\right)^{2/3} \approx \frac{456}{(V\mu_e)^{2/3} m_1^{1/3}}. \tag{4.29}$$

For a typical carbon-oxygen white dwarf, we have $\mu_e = 2$ and putting $V = 1$, we find $\eta_D > 287.3$ from equation (Eq. (4.29)) at center. Let r in equation (Eq. (4.21)) be the average separation of any two electrons at the center with $\rho_c \sim 10^{10}$ gm/cc, we obtain $\hbar^2/(m_e c r)^2 \sim \eta_D$. This justifies the noncommutativity under consideration to be spin-dependent curvature induced noncommutativity, as discussed previously below equation (Eq. (4.21)). Moreover, this suggests that as density decreases, separation r increases and hence, momentum space tends to become commutative.

The case of only lowest Landau level (m=0) filled up - Discussion :

As discussed previously, the origin of η can be traced back to a curvature effect arising in the gravitational interaction between the electrons. In fact in [77], a similar interpretation of momentum NC has been provided. The essence of the argument is that whenever momenta do not commute, a basic length scale must exist, or, to put it another way, the commutator dictates a scale. The inter-electron spacing in the current setup provides this length scale. As a result, this scale determines the momentum NC parameter η through Eq. (4.21).

Although the temperatures of white dwarfs are in the range of a few thousand Kelvins, the associated energy scales are still well below the Fermi level: $k_B T << E_F$. This also implies $k_B T << 2\eta c^2$ because for sufficiently strong values of η, which is the case here, the energy spacing becomes comparable/greater than electron rest mass energy and the electrons become relativistic i.e., $2\eta c^2 \geq m_e^2 c^4$. Thus we have,

$$\eta_D = \frac{\eta}{m_e^2 c^2} \approx 287 \tag{4.30}$$

which is quite large as compared to unity. Thus, this necessarily reiterates that the electrons are ultra-relativistic in our case and this is of utmost interest for our present purposes, as because this case only corresponds to existence of mass-limit of white dwarfs, even in Chandrasekhar's standard approach. Now, note that each level specified by m and with a definite Fermi momentum p_F has a degeneracy factor (D) of $\frac{\pi\eta}{h^2}$ per unit area, making the system to be a highly degenerate

Fermi system. The chief contribution towards the Fermi energy comes from η and so the factor $[\frac{{\epsilon_F}^2-1}{2\eta_D} = m_{max}]$ becomes of order unity as evident from Eq. (4.30). For our case, this factor is: $D \sim 10^{27}$ and from the density considered here, we have roughly the same order of number of electrons per unit area. This ensures that almost all the electrons will be contained in the lowest Landau level. The current situation is conceptually identical to that of a material sample subjected to low temperatures and intense magnetic fields; all of the electrons are contained within the first few Landau levels. It is in this regime that one encounters the celebrated quantum Hall effect, which is the defining property of Hall quantization. This arises mostly as a result of the large degeneracy of each Landau level.

4.3.1 Mass limit of white dwarfs

Finally, we now move on to see the implication of this increased degeneracy pressure on the hydrostatic balance condition of the white dwarf stars. We have computed the mass of these white dwarfs using the conventional Lane-Emden formalism. To that end, we now consider (Eq. (4.26)) to re-write it as,

$$P = K_{nc}\rho^{(1+\frac{1}{n})} \tag{4.31}$$

where $n = 3$ is the polytropic index. Following the Lane-Emden formalism (see Appendix 2 for details), the mass of white dwarfs for EoS given by the above equation (Eq. (4.31)) can be computed as,

$$M = \int_0^R 4\pi r^2 \rho dr = 4\pi a^3 \rho_c I_n, \tag{4.32}$$

where density $\rho = \rho_c \theta^n$ and radius $r = a\xi$ are expressed in terms of dimensionless variables θ and ξ respectively. Note that here ρ_c is the central density of the white dwarf, and a is given by

$$a = \sqrt{\frac{(n+1)K_{nc}\ \rho_c^{(1-n)/n}}{4\pi G}}$$

Furthermore, the radius R of the star is defined as $R = a\xi_1$ when at $\xi = \xi_1$, $\theta = 0$. The integral I_n in Eq. (4.32) has been defined as,

$$I_n = \int_0^{\xi_1} \theta^n \xi^2 d\xi \tag{4.33}$$

For $n = 3$, $I_n = 2.02$ and substituting the value of K_{nc} from (Eq. (4.26)) in (Eq. (4.32)) , we get

$$M = \left(\frac{hc}{G}\right)^{\frac{3}{2}} \frac{m_1^{1/2}}{\pi\mu_e^2 m_n^2} I_n = 4.68 M_\odot = M_{max} \tag{4.34}$$

Therefore, the mass-limit, turns out to be significantly super-Chandrasekhar, when the mass becomes independent of ρ_c, for most values of m_1 ($0 \lesssim m_1 < 1$), specifically for any $m_1 \gtrsim 0.1$. Thus, it allows for a range of the limiting mass from the standard $1.44M_\odot$ all the way upto $4.68M_\odot$ depending on the filling fraction m_1. Here we have taken $\mu_e = 2$ which is typically the case for carbon-oxygen white dwarfs.

Before ending up, we make certain observations in connection with the problem. Firstly, the m=0 Landau level only filled up case provides the strongest noncommutative contribution in the evaluation of the modified white dwarf mass. Additionally, by making this assumption, this problem could be solved exactly and analytically without having to resort to any numerical computation, thereby shedding light on the general qualitative aspects that can be expected in the current situation of momentum noncommutativity examined in the study. Secondly, it needs to be mentioned that our analysis is not a fully general relativistic (GR) treatment of white dwarfs at all. This is because when considering the hydrostatic balance equation (see Appendix 2 Eq. (6.11)) for the stellar structure that gives rise to the Lane-Emden equation (Eq. (6.15)), we have ignored the effects of GR. By addressing it in a completely generic relativistic framework, however, there are negligible departures in the mass-limit, as has been demonstrated in [116].

4.4 Remarks

In this chapter, we have first considered momentum noncommutativity and studied its consequences on the quantum mechanical statistical properties of an ensemble of identical and indistinguishable electrons in a three dimensional space. We found a modified spectrum of relativistic electrons adhering momentum noncommutativity which leads to an altered density of available states for these electrons. This in turn gave rise to an increased degeneracy pressure. A plausible implication of this altered dynamics of the electrons was discussed for white dwarfs by arguing that the gravitational field produced inside the stellar material should naturally introduce an effective noncommutativity in momentum space. This noncommutativity can be regarded as a trade of with the curvature effect arising from background gravity in the spirit of [15]. The increased pressure inside the star due to the augmented constant K_{nc} (Eq. (4.26), Eq. (4.27)) enabled it to withstand a larger mass inside itself. This scenario allows for an accessible mass range from Chandrasekhar mass limit $1.44M_{\odot}$ all the way upto $4.68M_{\odot}$. We therefore conclude this chapter by stating that the idealistic Chandrasekhar mass limit might get enhanced under the scenario considered here, thus providing a viable theoretical explanation for the recent findings of over-luminous unusual supernova SNeIa.

Chapter 5

Space-time noncommutativity: Kappa-Minkowski

In this final chapter, we will discuss the case of deformation in space-time, where time coordinate does not commute with spatial coordinates but the spatial coordinates will be commuting among themselves. Therefore, to illustrate a possible mechanism of appearance of such kind of space-times, we consider the most simplest case of 1+1 dimensions. A quantum theory of Moyal space-times in 1+1 dimensions has been recently constructed recently in [117]. In particular, here we will start with non-relativistic version of Kappa-Minkowski space-time. This space-time is characterized by a Lie-algebraic kind of deformation of space-time commutative algebra. This space-time originally emerged [118] as a candidate theory for describing special relativity in the presence of a minimal fundamental length scale. Since, this theory involves another fundamental constant of Nature which is the minimal length scale apart from the speed of light c, this has also been referred to as "doubly special relativity" [46]. Quantum descriptions of fields and particles in such space-times have already been extensively carried out [119–121] and has been obtained from quantum deformations of Poincare algebra [122]. In this thesis, by utilizing the idea of arbitrary time-reparametrizations and subsequently following Dirac's constraint analysis [63], we derive the Kappa-Minkowski space-time commutation relations in both nonrelativistic and relativistic cases from first principles, as a gauge-fixed version of a gauge theory. In the nonrelativistic context, this essentially requires one to enlarge the configuration space by including the time coordinate also as a configuration space variable, along with the spatial degrees of freedom. This has an innate level of conceptual similarity with relativistic notions of space and time and therefore the same construction can be generalized for the case of a relativistic free particle as well which we also demonstrate here.

5.1 Nonrelativistic free particle in Kappa-Minkowski using time-reparametrizations

In this section, we will study the classical dynamics of a free particle by using Dirac's theory of constraints [63]. Let us start by considering the non-relativistic action in an arbitrary potential as,

$$S[x(t)] = \int_{t_1}^{t_2} dt L\left(x, \frac{dx}{dt}\right), \quad L = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 - V(x,t). \tag{5.1}$$

Now under re-parametrizations of the time $t = t(\tau)$, where τ is the new evolution parameter which can be chosen in an arbitrary manner, except to just require only that $t = t(\tau)$ be a monotonically increasing function of τ and along with $x = x(\tau)$, we treat both of them as configuration space variables to re-write the action as,

$$S[x(\tau), t(\tau)] = \int_{\tau_1}^{\tau_2} d\tau L_\tau(x, \dot{x}, t, \dot{t}), \tag{5.2}$$

where,

$$L_\tau(x, \dot{x}, t, \dot{t}) = \dot{t} L\left(x, \frac{\dot{x}}{\dot{t}}\right) = \frac{1}{2}m\frac{\dot{x}^2}{\dot{t}} - \dot{t}V(x,t) \tag{5.3}$$

and, $\dot{t} = \frac{dt}{d\tau}$, $\dot{x} = \frac{dx}{d\tau}$. Therefore, we now have a reparametrization-invariant (RI) action for nonrelativistic mechanical systems. The two actions are equivalent in the sense that the Lagrangian equations for the functions $x(\tau), t(\tau)$ following from (Eq. (5.2)), imply the correct equations for $x(t)$ in Eq. (5.1) [123]. Now, the canonical momenta corresponding to the configuration space variables

$x(\tau)$ and $t(\tau)$ are defined as respectively:

$$p_x = \frac{\partial L_\tau}{\partial \dot{x}} = m\frac{\dot{x}}{\dot{t}} = m\left(\frac{dx}{dt}\right) \tag{5.4}$$

and

$$p_t = \frac{\partial L_\tau}{\partial \dot{t}} = -\frac{1}{2}m\frac{\dot{x}^2}{\dot{t}^2} - V(x,t) = -\frac{p_x^2}{2m} - V(x,t) = -H \tag{5.5}$$

Thus from above, we arrive at the following primary constraint -

$$\phi_1 = p_t + \frac{p_x^2}{2m} + V(x,t) \approx 0. \tag{5.6}$$

where ≈ 0 is equality in the weak sense [63]. Also, the canonical Hamiltonian is given by -

$$H_c = p_t\dot{t} + p_x\dot{x} - L_\tau = \dot{t}\phi_1 \approx 0. \tag{5.7}$$

The fact that this canonical Hamiltonian weakly vanishes is a generic feature of all time re-parametrized theories. Therefore, the total Hamiltonian of the system should be written as

$$H_T = H_c + \lambda\phi_1 \tag{5.8}$$

where λ is taken as a Lagrange multiplier. Note that there are no more constraints in the theory i.e. no secondary constraints in the system as,

$$\dot{\phi}_1 = \{\phi_1, H_T\} = 0 \tag{5.9}$$

Thus ϕ_1, being the only constraint, is also first-class. This constraint is being generally referred to as the Hamiltonian constraint for a time-reparametrized invariant theory and this in turn generates the arbitrary time-reparametrizations ($\tau \to \tau'$) [124, 125] as the first class constraints act as the generators of the gauge transformation. It is always possible to fix the gauge symmetry by imposing a gauge condition. Note that the space-time coordinates $x(\tau), t(\tau)$ transforms as a scalar under time reparametrisation:

$$\text{Under,} \quad \tau \to \tau' = \tau - \epsilon \tag{5.10}$$

$$x'^{\mu}(\tau') = x^{\mu}(\tau); \quad \mu = 0, 1 \tag{5.11}$$

Consequently, under an infinitesimal reparametrisation transformation, the infinitesimal change in the space-time coordinate is given by -

$$\delta x^{\mu}(\tau) = \epsilon \frac{dx^{\mu}}{d\tau} \tag{5.12}$$

Classically, if $\mathcal{G}$ is the generator of the infinitesimal transformation then for any phase space variable F, we have

$$\delta F = \{F, \mathcal{G}\}_{\text{PB}}. \tag{5.13}$$

Also, F is invariant if and only if $\delta F = 0$. And, we indeed have here,

$$\delta x^{\mu}(\tau) = \{x^{\mu}, \mathcal{G}\}_{\text{PB}} \tag{5.14}$$

where the generator $\mathcal{G}$ of time reparametrization transformation using Noether's theorem turns out to be, $G = \epsilon \dot{t} \phi_1$. Thus, time-reparametrization can be thought of as a gauge transformation [63,124].

Let us now consider only the free particle case i.e. $V(x,t) = 0$. Thus, we have now in place of (Eq. (5.5)),

$$p_t = -\frac{p_x^2}{2m} = -H_0 \tag{5.15}$$

where, H_0 denotes the usual free particle Hamiltonian.

5.1.1 Gauge-fixing and Dirac brackets

Note that here the free particle Hamiltonian, $H_0 = \frac{p_x^2}{2m}$ is gauge invariant because $\{H_0, \phi_1\}_{PB} = 0$. We would like to point out here that this holds only in the case of a free particle. Let us now consider the following gauge fixing condition for a free particle as,

$$\phi_2 = t - \tau - f(x, p_x) \approx 0. \tag{5.16}$$

Now, once we fix a gauge such as in ϕ_2 above, ϕ_1 and ϕ_2 becomes a pair of second class constraints as $\{\phi_1, \phi_2\}_{PB} \neq 0$. Let us then calculate the aforesaid Poisson bracket for the gauge fixing condition Eq. (5.16) -

$$\begin{aligned} \{\phi_1, \phi_2\}_{PB} &= \{p_t + H_0, t - \tau - f(x, p_x)\} \\ &= -1 + \frac{\partial f}{\partial x}\frac{\partial H_0}{\partial p_x} \\ &= -1 + \frac{p_x}{m}\frac{\partial f}{\partial x} \equiv C_{12}. \end{aligned} \tag{5.17}$$

The constraint matrix is constructed in the conventional manner as,

$$C_{\alpha\beta} = \begin{pmatrix} 0 & -1 + \frac{p_x}{m}\frac{\partial f}{\partial x} \\ 1 - \frac{p_x}{m}\frac{\partial f}{\partial x} & 0 \end{pmatrix}, \tag{5.18}$$

and its inverse,

$$(C^{-1})_{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{1 - \frac{p_x}{m}\frac{\partial f}{\partial x}} \\ \frac{1}{-1 + \frac{p_x}{m}\frac{\partial f}{\partial x}} & 0 \end{pmatrix} \equiv C^{\alpha\beta}. \tag{5.19}$$

Finally, the Dirac bracket is defined for any pair of arbitrary phase space variables M and N by -

$$\{M, N\}_{DB} = \{M, N\}_{PB} - \{M, \phi_\alpha\}_{PB} C^{\alpha\beta} \{\phi_\beta, N\}_{PB}, \tag{5.20}$$

where $C^{\alpha\beta}$ is the inverse matrix defined in Eq. (5.19). Now in the gauge fixing condition, choosing $f = -axp_x$ (where a is a constant parameter) yields the following Dirac brackets after a straight forward calculation -

$$\{t, x\}_{DB} = \frac{ax}{1 + 2aE} \quad ; \quad E = \frac{p_x^2}{2m}. \tag{5.21}$$

Similarly,

$$\{t, p_t\}_{DB} = \frac{2aE}{1 + 2aE}, \tag{5.22}$$

$$\{p_x, t\}_{DB} = \frac{ap_x}{1 + 2aE}, \tag{5.23}$$

and

$$\{x, p_x\}_{DB} = \frac{1}{1 + 2aE}. \tag{5.24}$$

Therefore, we essentially have

$$\{t, x\}_{DB} = \lambda x \tag{5.25}$$

where $\lambda = \frac{a}{1+2aE}$ is a constant as H_0 is gauge-invariant for the case of the free particle (since $\{H_0, \phi_1\}_{PB} = 0$). Thus, we have derived the kappa-Minkowski Poisson bracket relation in the framework of a time-reparametrized theory. However, it should be noted that the constant parameter appearing in the defining commutation relation of Kappa-Minkowski spacetime Eq. (5.25) has been found to be energy dependent in our case. In the limit $a \to 0$, we get back the undeformed results as in the standard classical mechanics, which will basically correspond to identifying the time coordinate with the evolution parameter Eq. (5.16). These phase-space variables can now be promoted to the level of quantum noncommuting operators acting on a Hilbert space, following from Dirac quantization prescription [63]. One can now proceed with the quantization of the theory by using various star product realizations of the Kappa-Minkowski algebra Eq. (5.25) as has been worked out in [126, 127], however, this is beyond the scope of the present chapter.

Now we want to compute the equation of motion of the free particle on the constraint surface in the κ-deformed space-time. This is now a second-class theory and the Dirac brackets imply a strong imposition of the second class constraints. Therefore, the equation of motion on the constraint surface for the free particle will be [128]:

$$\dot{x} = \frac{dx}{d\tau} = \frac{\partial x(\tau)}{\partial \tau} + \{x, H_0\}_{DB} \quad , \qquad \dot{t} = \frac{dt}{d\tau} = \frac{\partial t}{\partial \tau} + \{t, H_0\}_{DB} \tag{5.26}$$

After a straight forward calculation, we find

$$\dot{x} = \frac{p_x}{m + ap_x^2} \quad , \quad \dot{t} = \frac{m}{m + ap_x^2}. \tag{5.27}$$

Additionally, we have -

$$\dot{p}_x = 0 \tag{5.28}$$

Therefore, the equation of motion for the free particle retains its form also in this deformed space-time structure :

$$\ddot{x} = 0 \tag{5.29}$$

In this context, it will be worthful to mention that in general in arbitrary potentials also, this procedure may give rise to non-standard symplectic structure among the phase-space variables and, in particular, will generate a deformed space-time algebra [50]. However, there should not be any deformation in the equation of motion of the particle as shown above.

5.2 Relativistic free particle

In the earlier section, we studied, in particular, the problem of non-relativistic free particle but here in this section, we take up the case of relativistic free particle in 1+1 dimension whose action is given by,

$$S_{rel} = -m \int d\tau \sqrt{-\dot{x}^{\mu}\dot{x}_{\mu}} \tag{5.30}$$

where $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$ and τ is proper-time or in general an affine parameter, the Minkowski metric $\eta_{\mu\nu}$ =diag$(-1,+1)$ and $\mu,\nu = 0,1$. Note that we have set the speed of light as $c = 1$. Now upon finding the canonical momenta, this action leads to the mass shell or Einstein constraint as given by,

$$\phi_1 = p^\mu p_\mu + m^2 \approx 0. \tag{5.31}$$

where p_μ are the canonical momenta conjugate to x^μ in the Lagrangian formalism and we therefore have $\{x^\mu, p_\nu\} = \delta^\mu_\nu$. Note that this action (Eq. (5.30)) is manifestly in time RI form and naturally both x^μ $(x^0 = t, x^1 = x)$ are treated as configuration space variables. Next, we take the gauge fixing condition as,

$$\phi_2 = t - \tau - f(x, p_x) \approx 0. \tag{5.32}$$

Like in the case of the previous section, the constraints ϕ_1 and ϕ_2 form a set of second-class constraints as,

$$\begin{aligned} \{\phi_1, \phi_2\} &= \{-p_t^2 + p_x^2 + m^2, t - \tau - f\} \\ &= -2p_t + 2p_x \frac{\partial f}{\partial x} \equiv C_{12}. \end{aligned} \tag{5.33}$$

Now the inverse of the constraint matrix $C_{\alpha\beta}$ is given by:

$$(C^{-1})_{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{2p_t - 2p_x \frac{\partial f}{\partial x}} \\ \frac{1}{-2p_t + 2p_x \frac{\partial f}{\partial x}} & 0 \end{pmatrix} \equiv C^{\alpha\beta}. \tag{5.34}$$

Now by using (Eq. (5.20)) and with the same choice of f as in previous section, $\underline{f = -axp_x}$, we get

$$\{t, x\}_{DB} = \frac{ax}{1 + aE'} \quad ; \quad E' = \frac{E^2 - m^2}{E} \tag{5.35}$$

Moreover we have,

$$\{t, p_t\}_{DB} = \frac{-aE'}{1 + aE'} \quad , \tag{5.36}$$

$$\{p_x, t\}_{DB} = \frac{ap_x}{1 + aE'} \quad , \tag{5.37}$$

and

$$\{x, p_x\}_{DB} = \frac{1}{1 + aE'} \tag{5.38}$$

where we have used $\{x^\mu, p^\nu\} = \eta^{\mu\nu}$. On explicitly retaining c till the end of the calculations, and then taking the limit $c \to \infty$, one does indeed get back the results of the previous section for the nonrelativistic free particle.

5.3 Remarks

The above demonstration provides a new dynamical realization of the Kappa-Minkowski space-time and/or algebra in both nonrelativistic as well as relativistic contexts. This scheme has been generalized for oscillatory potentials in nonrelativistic case in [50], where we have worked this out explicitly. It is to be noted that we have dealt with non-stationary constraints (Eq. (5.16)) in our Hamiltonian analysis. So they are not true second-class constraints in the Dirac sense. However, this did not stop us from carrying out Dirac's analysis as has been pointed out in the rigorous formulation of Tyutin and Gitman for dealing with such kind of constraints [124,129]. In regard to this issue, one can also naively assume that x and p_x have no explicit τ dependence in the gauge-fixing relation [128,130]. Also, it is quite encouraging to find that the same non-trivial gauge fixing condition yields relativistic and non-relativistic Kappa-deformed space-time algebras, despite the fact that the corresponding actions have different properties with respect to time-reparametrization transformations - the former is RI by default whereas the non-relativistic action needs to be cast in a RI form.

Chapter 6

Conclusion

Ever since quantum mechanics and general theory of relativity, which are the two cornerstones of modern day physics has come into existence, their consistent amalgamation has been the holy grail of theoretical physicists. With more than seven decades of research, a quantum theory of gravity has still remained a hard nut to crack. However, as is customary with physics and sciences in general, this journey has seen a confluence of ideas, cutting across different areas of theoretical physics and establishing links between apparently unrelated aspects of those areas which no one could anticipate earlier. Examples include the relations between noncommutative geometry, statistical mechanics, quantum information science, non-linear dynamics and chaos theory. Novel relations have emerged between black hole physics, which is the most natural laboratory for testing quantum gravity phenomena, and condensed matter physics using gauge/gravity dualities. All the candidate theories of quantum gravity seems to suggest that the spacetime as we perceive it and which is modeled as a pseudo-Riemannian differentiable manifold has to be replaced by some sort of quantum spacetime, having an intrinsic minimum length scale associated with it. For example, in loop quantum gravity it was found to be given by some spin networks, where each of the nodes represent "a quantum of space". It was further shown that area (respectively volume) of surfaces (respectively regions) are quantized, where the above mentioned minimal length scale naturally appears. On the other hand, in noncommutative geometry, the quantum spacetime is taken to be given by some noncommutative algebra generated by spacetime coordinates, which are now elevated to the level of operators. In other words, this noncommutative algebra is obtained by suitably deforming the commutative algebra, where again the deformation parameter is given in terms of this minimal length scale. This, we feel, is a simpler and promising route to capture the minimal length scale physics. Also, the approach of noncommutative geometry could be applied to provide a new framework for the Standard model in particle physics albeit at the classical level and to go beyond and obtain the grand unification of elementary particles ($SU(2)_R \otimes SU(2)_L \otimes SU(4)$ gauge theory) as proposed by Pati and Salam. Not only that, it has also become relevant from the condensed matter physics perspective. In the framework of Noncommutative geometry, a rigorous formulation of Hall plateaus in Integer Quantum Hall effect can be given. This geometry has also successfully described many aspects of the fractional quantum Hall effect (FQHE). Thereafter it was realized that noncommutative geometry is a good setting for capturing anyonic effects where the noncommutative parameter is directly proportional to the fractional spin of particles (anyonic spin). Therefore, the primary aim of this thesis has been to look for theories of noncommutative geometry which yields interesting novel features at the quantum level and also to interrelate noncommutative theories with their commutative counterparts. The results derived in this thesis thus are expected to have implications and find relevance both in Planck scale physics as well as in condensed matter physics and quantum-optics with each having their associated length scales.

In the first chapter "Introduction", we have presented the idea behind quantum space-time theories and the associated framework of noncommutative geometry. This also provides a general background setting for the considerations taken up in rest of the thesis.

In the second chapter, we consider the phenomenon of dissipation in deformed spaces. Both, dissipation and noncommutative geometry have been known to violate time-reversal symmetry. For this study, we followed Bateman's doubling procedure in the so-called indirect representation in an ambient two-dimensional noncommutative Moyal plane. This is a very peculiar system in its own right as the Hamiltonian of the system turns out to have no lower bound and therefore hinders standard canonical quantization. In order to circumvent this issue, we considered extra interactions in our parent system. We carried out the quantization in the Moyal plane using Hilbert-Schmidt operator formulation using coherent states. Consequently, we performed the canonical analysis too. At the final stage, we could remove the additional interactions to get the Bateman oscillators. We found out that there is an extra noncommutative contribution towards the damping. Thus, usual simple harmonic oscillators can give rise to a damping in presence of deformation. Alternatively, we can eliminate the damping/anti-damping part of the damped harmonic oscillator (DHO) in noncommutative scenario making it a perfectly integrable one. Therefore, we have established in some sense a duality between these previously unrelated aspects, namely deformation in symplectic structure and dissipation. Moreover, as a by-product of our analysis, through the inclusion of additional interactions in this problem, we have computed the energy spectrum of the problem of a anisotropic harmonic oscillator in two dimensions in the presence of a background field. Our results match with that from t' Hooft's scheme, for quantizing the DHO, in the commutative limit.

We then moved on to study entropic properties of deformed systems in the third chapter. This consideration has been hinted by t' Hooft's arguments that unbounded Hamiltonian systems lead to an information loss. Now such systems can be shown to exhibit noncommutative geometry in the position coordinates wherein the final effective Hamiltonian indeed has a lower bound. Therefore as a prototype model, we considered the exotic oscillators. We computed the von-Neumann entropy and found that there is a non-zero entropy in the system due to existence of the symplectic deformation (noncommutativity). We generalized our problem to consider oscillators in two-dimensional noncommutative phase-space. Here also, a non-zero entanglement entropy is obtained and the vacuum of the noncommutative theory could be expressed as a two-mode squeezed vacuum with respect to the commutative vacuum. Hence, this brought out new relations between commutative and noncommutative theories. A comparison is then made with squeezing in three-wave mixing process. Lastly, we mapped the exotic oscillator problem to the well known Landau problem in presence of a harmonic potential and we revealed here a Unruh-like physics in the aforesaid Landau dynamics.

In the fourth chapter, we considered solely momentum space noncommutativity. Basically, this can arise due to the presence of position space curvature. Physically this kind of noncommutativity is different from coordinate space noncommutativity and we make it explicit by computing the energy of a free relativistic electron moving in a three-dimensional space which exhibits momentum noncommutativity by solving the Dirac equation. Unlike in the case of position noncommutativity, here, there is a deformation in the spectrum of the free particle. After computing the spectrum, we study the consequences of this deformation in quantum mechanical statistical properties. The pressure equation gets modified and we try to find its implications in the matter of white dwarf stars, where we argue that such momentum noncommutative structures can emerge due to the effects of small-scale curvature inside white dwarfs. Using Lane-Emden formalism we arrived at a higher mass-limit of these stars, $4.68M_{\odot}$. Basically, this is due to the increase in electron degeneracy pressure under the above-mentioned consideration, which enables the star to hold a greater mass. This analysis provides a plausible theoretical explanation for the observations of some over-luminous peculiar SNeIa which seem to indicate a higher mass of the progenitor white dwarfs.

In the fifth chapter, we considered deformations in space-time algebra and investigated the non-relativistic Kappa-Minkowski space-time in (1+1) dimensions. By taking up the case of a free particle, we deduced the Kappa-Minkowski deformations using time-reparametrizations and following Dirac's analysis of constraints. Our derivation is from first principles and this procedure may be adopted for the case of harmonic oscillators, other conservative potentials, etc. Furthermore, this can also be generalized for the case of a relativistic free particle by taking advantage of the reparametrized-invariant form of the relativistic action easily.

In future, I would like to investigate Unruh effect in quantum space-time, say in the Moyal case as well as for Kappa-Minkowski kind. This will be interesting to compare the results. However, that is far from near as this will involve concepts out of the scope of the present thesis. Further, I am also interested to consider dissipative field theory in noncommutative space and study the entanglement entropy.

Appendix 1

Here, we are going to provide a brief description regarding the connection between single-mode and two-mode squeezings, well-studied in the field of Quantum Optics. Let us consider the relation between the single-mode non-commutative annihilation operators $a_i(\theta,\eta)$ and single-mode commutative annihilation operators b_j (Eq. (3.42)). This transformation can be realized as a squeezing transformation with the corresponding single-mode squeezing operator being given by,

$$S^{(1)}_{b_j}(s) = e^{-\frac{1}{2}(s b_j^2 - s^* b_j^{\dagger 2})} \;\; ; \; j = 1, 2. \tag{6.1}$$

Here the subscript 1 corresponds to single-mode squeezing. The complex squeezing parameter s is given by

$$s = \phi e^{i(\psi+\xi)} \quad ; \qquad \phi = \frac{1}{4}\log\left(1 + 2\lambda^2 + \frac{m^2\omega^2\theta^2}{4\hbar^2} + \frac{\eta^2}{4m^2\omega^2\hbar^2}\right) \tag{6.2}$$

and ψ, ξ previously defined in Eq. (3.43). In fact using the disentanglement theorem of $SU(1,1)$ [131], it can be shown that-

$$a_j(\theta,\eta) = S^{(1)}_{b_j}(s)\; b_j \; (S^{(1)}_{b_j}(s))^\dagger \tag{6.3}$$

Finally, the relation (Eq. (3.47)) between the normal-mode annihilation operators $(a_\pm(\theta,\eta))$ and commutative Fock operators can be realized by the following two-mode complex squeezing transformation:

$$\begin{pmatrix} a_+(\theta,\eta) \\ a_-(\theta,\eta) \end{pmatrix} = \begin{pmatrix} e^{i\psi}\left[cosh(\phi) b_+ - sinh(\phi) e^{-i(\psi+\xi)} b_-^\dagger\right] \\ ie^{i\psi}\left[cosh(\phi) b_- - sinh(\phi) e^{-i(\psi+\xi)} b_+^\dagger\right] \end{pmatrix} \tag{6.4}$$

In other words, this is a complex Bogolyubov transformation corresponding to the action of the two-mode squeezing operator $S^{(2)}_{(b_+,b_-)}(s) = e^{-(sb_+b_- - s^* b_+^\dagger b_-^\dagger)}$ i.e.

$$\begin{pmatrix} a_+(\theta,\eta) \\ a_-(\theta,\eta) \end{pmatrix} = S^{(2)}_{(b_+,b_-)}(s) \begin{pmatrix} b_+ \\ ib_- \end{pmatrix} (S^{(2)}_{(b_+,b_-)}(s))^\dagger \tag{6.5}$$

with the complex squeezing parameter given by $s = \phi e^{i(\psi+\xi)}$, which is the same as that in the single mode squeezing case. It can be shown easily that :

$$S^{(2)}_{(b_+,b_-)}(s) = U_{B'} \left(S^{(1)}_{b_+}(s) \otimes S^{(1)}_{b_-}(s) \right) U_B \tag{6.6}$$

where U'_B and U_B are the unitary operators corresponding respectively to the lossless beam splitter matrices:

$$B' = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} \;\; ; \;\; B = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{6.7}$$

Thus we see from (Eq. (6.6)) that any two-mode squeezing transformation is related to a single-mode squeezing transformation - with the same amount of squeezing – via the actions of two fixed 50:50 lossless beam splitters. As lossless beam splitters can neither create nor can it destroy non-classicality (in terms of photon number statistics), therefore, the actual sources of the non-classical operation of squeezing here are the position and momentum observables non-commutativity parameters θ and η respectively. In fact, it follows directly from the aforesaid expression of the squeezing parameter that it's value is 0 if and only if $\theta, \eta = 0$. The non-classicality in two-mode squeezing is actually generated by the non-classicality present in the single-mode squeezing with complex squeezing parameter $s = \phi e^{i(\psi+\xi)}$. Now, in case of only position noncommutativity $\eta = 0, \theta \neq 0$, we also have a squeezing transformation as in (Eq. (6.3)) with a real squeezing parameter given by $s = \frac{1}{4}\log\left(1 + \frac{m^2\omega^2\theta^2}{4\hbar^2}\right)$.

It may be recalled here that corresponding to the action of any lossless beam splitter with associated (unitary) beam splitter matrix,

$$B(\Theta,\Phi,\Psi)=\begin{pmatrix}\cos\frac{\Theta}{2}e^{i(\Psi+\Phi)/2} & \sin\frac{\Theta}{2}e^{i(\Psi-\Phi)/2}\\ -\sin\frac{\Theta}{2}e^{-i(\Psi-\Phi)/2} & \cos\frac{\Theta}{2}e^{-i(\Psi+\Phi)/2}\end{pmatrix}, \tag{6.8}$$

its unitary representation $U_{B(\Theta,\Phi,\Psi)}$ is given by (with input mode operators a_1, a_2) :

$$U_{B(\Theta,\Phi,\Psi)} = e^{-i\Phi L_3}e^{-i\Theta L_2}e^{-i\Psi L_1} \tag{6.9}$$

where, $L_1 = \frac{1}{2}(a_1^\dagger a_2 + a_2^\dagger a_1); L_2 = \frac{1}{2i}(a_1^\dagger a_2 - a_2^\dagger a_1); L_3 = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)$ are the three generators of su(2) algebra.

Appendix 2

We provide here a brief discussion of Lane-Emden formalism which is generally utilised to solve for the structure of a star that satisfies the polytropic pressure relation as in Eq. (4.31). The conservation of mass (i.e. equation of continuity) gives -

$$\frac{dM}{dr} = 4\pi r^2 \rho \tag{6.10}$$

and, the hydrostatic balance equation gives,

$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho \tag{6.11}$$

Differentiating both sides of Eq. (6.11) and subsequently using (Eq. (6.10)), we get

$$\frac{d}{dr}\left(\frac{1}{\rho}\frac{dP}{dr}\right) = -\frac{2}{\rho r}\frac{dP}{dr} - 4\pi G\rho \tag{6.12}$$

Now, using the polytropic equation of state: $P = K_{nc}\rho_c^{(1+\frac{1}{n})}\theta^{n+1}$ (Eq. (4.31)) where, $\rho = \rho_c\theta^n$ we have from above -

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 K_{nc}\rho_c^{\frac{1}{n}}(n+1)\frac{d\theta}{dr}\right) = -4\pi G\rho_c\theta^n \tag{6.13}$$

Putting $r = a\xi$, where ξ is dimensionless, and the constant a has the dimensions of length and is chosen as -

$$a = \sqrt{\frac{(n+1)K_{nc}\ \rho_c^{(1-n)/n}}{4\pi G}}\ , \tag{6.14}$$

we can cast the above in the following form:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \tag{6.15}$$

Thus we arrive at the useful Lane-Emden equation. As this is a 2nd-order differential equation, we require two boundary conditions to integrate it which are as follows:

$$\theta(\xi = 0) = 1 \quad \text{and} \quad \left(\frac{d\theta}{d\xi}\right)_{\xi=0} = 0 \tag{6.16}$$

where the second condition follows from the argument that there should be no kink in density function at the centre of star where it is maximum. Now, the mass of the star will be given by Eq. (6.10) -

$$M = \int_0^R 4\pi r^2 \rho \; dr = 4\pi a^3 \rho_c \int_0^{\xi_1} \; \xi^2 \theta^n d\xi \tag{6.17}$$

On using Eq. (6.15) and the associated boundary conditions Eq. (6.16), we can re-write the mass of the star as,

$$M = -4\pi a^3 \rho_c \; \xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_1} \tag{6.18}$$

Note that with increasing radius, density of the star decreases and hence, $\frac{d\theta}{d\xi} < 0$. On solving the Lane-Emden equation (Eq. (6.15)) numerically with the polytropic index n=3, we have $\xi_1^2|\theta'(\xi_1)| = 2.02$.

Substituting the above numerical value and using Eq. (4.27),Eq. (6.14), we finally deduce the expression of mass of the star as in Eq. (4.34).

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